

# On the Derivation of the Monodromy Matrix of the Buck Converter

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**Abstract** - The study of bifurcations phenomena in the dc-to-dc buck converter, which may not be common in smooth dynamic systems, is based on the derivation of an iterative map that approximates the operation of the converter. This paper presents a quantitative proof for the derivation of the total fundamental solution matrix before, during and after the system changes topologies as a result of the converter switching. The proof is based on upper semi-continuous Filippov inclusions applied on transversal intersections of the discontinuous hypersurface.

**Index Terms** - Buck converter, bifurcation analysis, Filippov inclusions, discontinuous systems.

## I. INTRODUCTION

The first qualitative studies of nonlinear power electronics models were presented in 1984 by Brockett and Wood [1] and were later continued by Hamill and Jeffries [2], who effectively triggered the study of what was referred to until then as ‘unknown’ or ‘unwanted’ instabilities. Hamill and Jeffries [2] modeled a switched mode power converter with an iterative map and the previously mentioned instability is shown to exist for a particular set of parameters. The loci of these parameters were determined by the Cobweb diagram of the map. Bifurcation and chaotic phenomena may appear in much simpler circuits, like a series RLD (resistor-inductor-diode) circuit with a nonlinear diode. A boost converter under current control has also been shown to exhibit chaotic patterns [3]. These bifurcation phenomena, referred to as border collision, are not common to other smooth systems and can only be found in nonlinear, non-smooth systems and in the power electronics case controlled by a pulse width modulator.

One way to study the bifurcation phenomena of periodic systems; like the buck converter; is to find the Poincare map and its eigenvalues that are identical to the Floquet multipliers of the systems, i.e. the eigenvalues of the total solution matrix. The problem with this approach is the calculation of the fundamental matrix of the buck converter, since the state space is separated into two (or more) parts from a hypersurface that describes the discontinuity of the vector field due to the switching of the inverter.

This paper presents a rigorous mathematical derivation of the previously mentioned fundamental solution matrix and hence through that it is possible to describe the state solutions before, after and, for the first time, during the crossing of the hypersurface defined by the switching discontinuities of the converter. The existence and uniqueness of the solutions are examined using methods first developed to study systems of differential equations with discontinuous right hand sides using differential inclusions [4 - 6].

## II. SYSTEM ANALYSIS

### A. Filippov Theory

A system is called continuous if its mathematical model can be described by a set of differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

where  $\mathbf{f}$  is a smooth vector field in a given domain. The existence and uniqueness of that system is guaranteed by the Lipschitz theorem [5] which states that when there exists a constant  $L$  such that

$$\|\mathbf{f}(\mathbf{x}_A, t) - \mathbf{f}(\mathbf{x}_B, t)\| \leq L \|\mathbf{x}_A - \mathbf{x}_B\| \quad (2)$$

and  $\mathbf{f}$  is linearly bounded, then (1) has a unique solution which depends only on the initial condition.

If the system is not continuous in time then a variation of the above theorem [4, 7] can guarantee a solution (referred to as a Caratheodory solution). But if the system is discontinuous in  $\mathbf{x}$  then the above methods cannot be applied. Instead a Filippov solution can now be defined [5] that uses set valued functions, i.e. functions that return a single vector when  $\mathbf{f}$  is continuous and the convex closure over the limits of the discontinuity when  $\mathbf{f}$  is discontinuous:

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t) = \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{co} \mathbf{f}(B(\mathbf{x}, \delta) - N, t) \quad (3)$$

where  $N$  are the sets that contain the points of discontinuity,  $\bigcap_{\mu V=0}$  is the intersection of all  $N$  sets with

Lebesgue measure zero,  $B(\mathbf{x}, \delta)$  is a ball of radius  $\delta$  and  $\overline{CO}$  denotes the smallest closed convex set. To insure existence, but not uniqueness, of the solution,  $\mathbf{F}$  has to be upper semi-continuous, closed, convex and bounded [5], (2) is generally referred to as Filippov Inclusion (FI).

### B Mathematical model of the buck converter

The open loop buck converter described by (4) and (5) is an autonomous, linear, piecewise system:

$$\frac{di}{dt} = \begin{cases} \frac{v_{in} - v}{L}, & S \text{ Conducting} \\ -\frac{v}{L}, & S \text{ Blocking} \end{cases} \quad (4)$$

$$\frac{dv}{dt} = \frac{i - v/R}{C} \quad (5)$$

It is obvious from (4) and (5) that there is a discontinuity when the main switching element  $S$  passes from conduction to blocking, since  $\frac{v_{in} - v}{L} \neq -\frac{v}{L}$  and  $v_{in} \neq 0$ . Normal

techniques for analyzing the stability of the converter are not applicable as small changes in the supply voltage may cause unacceptable output behavior due to the discontinuous characteristics of the system [8 & 9]. Various closed loop control techniques are therefore applied to compensate for possible voltage changes. Fig. 1 shows such a scheme with a closed loop proportional voltage controller.

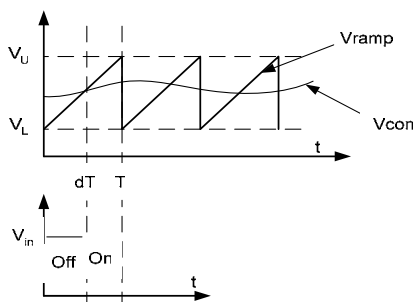
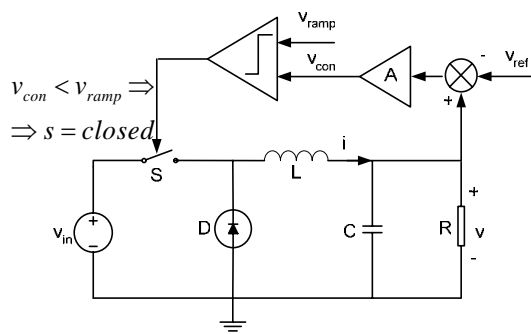


Fig. 1 Closed loop, voltage controlled buck dc-dc converter

The buck converter shown in Fig. 1 has been simulated using the following data:  $L=20\text{mH}$ ,  $R=22\Omega$ ,  $C=47\mu\text{F}$ ,  $A=8.4$ ,  $V_L=3.8\text{V}$ ,  $V_U=8.2\text{V}$ ,  $v_{in}=24\text{V}$ ,  $v_{ref}=12\text{V}$ . Fig. 2 shows output voltage, inductor current, control and ramp signals.

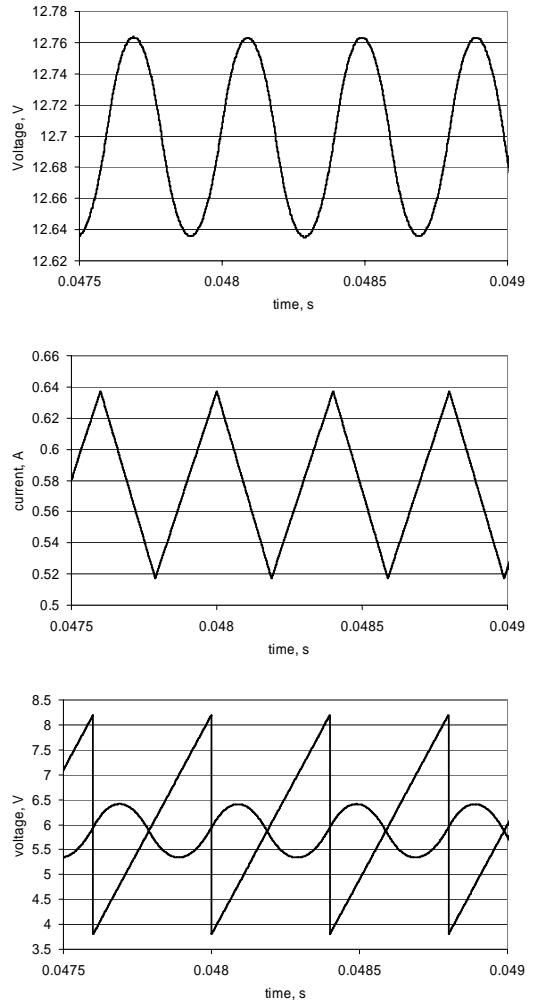


Fig. 2 Simulated output voltage, inductor current, control and ramp signal

Under normal conditions the output of the converter will be a periodic signal with a mean value close to the desired voltage and a period that is equal to the period of the PWM signal. If the input voltage is increased above a specific value [3], the system will change its frequency characteristics to period 4, 8 and at some point due to a border collision (or crisis) the system will enter into a chaotic region.

In this paper, the total fundamental solution matrix (also called monodromy matrix) of the system before, after and during the discontinuous hypersurface is calculated. The system will be shown to have discontinuous and not just non-smooth vector fields and the existence and uniqueness of the solution will be analyzed by studying the system as a Filippov Inclusion [5].

### III. EXISTENCE AND UNIQUENESS

The closed loop equations of the buck converter are given by:

$$\frac{di}{dt} = \begin{cases} \frac{v_{in} - v}{L}, A(v - v_{ref}) < v_{ramp} \\ -\frac{v}{L}, A(v - v_{ref}) > v_{ramp} \end{cases} \quad (6)$$

$$\frac{dv}{dt} = \frac{i - v/R}{C} \quad (7)$$

Defining  $x_1 = v$  and  $x_2 = i$  result in:

$$\frac{dx_2}{dt} = \begin{cases} \frac{v_{in} - x_1}{L}, A(x_1 - v_{ref}) < v_{ramp} \\ -\frac{x_1}{L}, A(x_1 - v_{ref}) > v_{ramp} \end{cases} \quad (8)$$

$$\frac{dx_1}{dt} = \frac{x_2 - x_1/R}{C} \quad (9)$$

The discontinuity can be represented by a hypersurface (h) that splits the state space into two parts:

$$h(\mathbf{x}(t)) = 0 \Leftrightarrow x_1 - v_{ref} - \frac{v_{ramp}}{A} = 0, A \neq 0 \quad (10)$$

$$v_{ramp} = V_L + (V_U - V_L) \left( \frac{t}{T} \text{mod}(1) \right) \quad (11)$$

where T is the time period. Equations 10 and 11 state that the hypersurface is non autonomous and this directly implies that the closed loop system is also non autonomous since the crossing of the state vector on the hypersurface will be explicitly depended on time (among others). Furthermore, since the stability properties can be derived by the monodromy matrix, only one cycle has to be studied, hence:

$$v_{ramp} = V_L + (V_U - V_L) \frac{t}{T} \quad (12)$$

When the switching occurs:  $v_{ramp} = V_L + (V_U - V_L)d$ ,  $d = \text{duty cycle}$

$$h(\mathbf{x}(t)) = 0 \Leftrightarrow x_1 - v_{ref} - \frac{V_L + (V_U - V_L)d}{A} = 0 \quad (13)$$

$$h(\mathbf{X}) = x_1 - K_1(t) = 0 \quad (14)$$

Hence the two dimensional state space is separated in three parts:

$$X_- \cup \Sigma \cup X_+ = R^2 \quad (15)$$

where

$$\begin{aligned} X_- &= \{ \mathbf{x} \in R^2 : h(\mathbf{x}(t)) < 0 \}, \\ X_+ &= \{ \mathbf{x} \in R^2 : h(\mathbf{x}(t)) > 0 \} \quad \text{and} \\ \Sigma &= \{ \mathbf{x} \in R^2 : h(\mathbf{x}(t)) = 0 \} \quad \text{and} \quad \mu_L(\Sigma) = 0 \quad (\text{i.e. zero Lebesgue measure}). \end{aligned}$$

(8) and (9) can be written as an upper semi continuous FI:

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t) = \bigcap_{\delta > 0} \bigcap_{\mu \neq 0} \overline{\text{co}}\{B(x, \delta) - N, t\} \quad (16)$$

(16) has a solution in the sense of Filippov if the vector field enters the hyper-surface  $\Sigma$  instantly. Since there is only one discontinuity:

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t) = \begin{cases} \mathbf{f}_-(\mathbf{x}, t), \mathbf{x} \in V_- \\ \overline{\text{co}}\{\mathbf{f}_-(t, \mathbf{x}), \mathbf{f}_+(t, \mathbf{x})\}, \mathbf{x} \in \Sigma \\ \mathbf{f}_+(\mathbf{x}, t), \mathbf{x} \in V_+ \end{cases} \quad (17)$$

$$\text{where } \mathbf{f}_-(\mathbf{x}, t) = \begin{bmatrix} \frac{x_2 - x_1/R}{C} \\ \frac{v_{in} - x_1}{L} \end{bmatrix}, \mathbf{f}_+(\mathbf{x}, t) = \begin{bmatrix} \frac{x_2 - x_1/R}{C} \\ -\frac{x_1}{L} \end{bmatrix} \quad (10)$$

I.e. at the switching hypersurface

$$\mathbf{f}_-(\mathbf{x}, t) \neq \mathbf{f}_+(\mathbf{x}, t) \Rightarrow \frac{v_{in} - x_2}{L} \neq -\frac{x_2}{L}, \text{ and therefore the}$$

system has discontinuous vector fields. The convex hull is defined as:

$$\overline{\text{co}}\{\mathbf{f}_-(\mathbf{x}, t), \mathbf{f}_+(\mathbf{x}, t)\} = \left[ \begin{array}{c} \frac{x_2 - x_1/R}{C} \\ \overline{\text{co}}\left\{ \frac{v_{in} - x_1}{L}, -\frac{x_1}{L} \right\} \end{array} \right] =$$

$$\left[ \begin{array}{c} \frac{x_2 - x_1/R}{C} \\ \left\{ (1-q) \frac{v_{in} - x_1}{L} - q \frac{x_1}{L} \right\} \end{array} \right], \forall q \in [0, 1]$$

The normal to the hypersurface is:

$$\mathbf{n} = \nabla h(\mathbf{x}) = \begin{bmatrix} \frac{\partial h(\mathbf{x})}{\partial x_1} \\ \frac{\partial h(\mathbf{x})}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (18)$$

Therefore the projections of  $\mathbf{f}_-$  and  $\mathbf{f}_+$  on  $\mathbf{S}$  are given by:

$$\mathbf{n}^T \mathbf{f}_- = \frac{x_2 - x_1/R}{C}, \quad \mathbf{n}^T \mathbf{f}_+ = \frac{x_2 - x_1/R}{C}$$

$$\text{Hence } \mathbf{n}^T \mathbf{f}_- \cdot \mathbf{n}^T \mathbf{f}_+ = \left( \frac{x_2 - x_1/R}{C} \right)^2 > 0,$$

i.e. a transversal intersection as shown in Fig. 3.

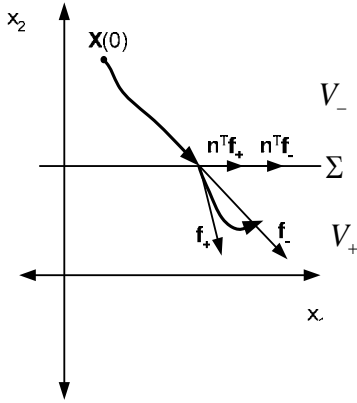


Fig. 3 A transversal intersection

A system that transversally intersects the hypersurface has a unique solution [6].

#### IV. CALCULATION OF THE OVERALL MONODROMY MATRIX

Assuming that the initial state vector starts at  $V_-$  ( $t = t_0$ ) then after some time ( $t = t_\Sigma$ ) (or  $t = dT$ ) the solution will transversally intersect the hypersurface  $\Sigma$ , as defined previously (Fig. 4). Before the intersection the system is smooth and therefore a fundamental matrix may be defined as:

$$\Phi(t_0, x_0, t_1) \text{ for } t_1 \in [0, t_\Sigma) \text{ or } t_1 \in [0, dT) \quad (19)$$

After the intersection another fundamental matrix may also be defined as:

$$\Phi(t_\Sigma, x_\Sigma, t) \text{ for } t \in (t_\Sigma, \infty) \text{ or } t \in (dT, T) \quad (20)$$

For the switched system, the monodromy matrix  $\Phi(t_0, x_0, t)$  for  $t \in (t_\Sigma, \infty)$  is given by:

$$\Phi(t_0, x_0, t) = \Phi(t_\Sigma, x_\Sigma, t) \mathbf{S} \lim_{t_1 \rightarrow t_\Sigma} \Phi(t_0, x_0, t_1) \quad (21)$$

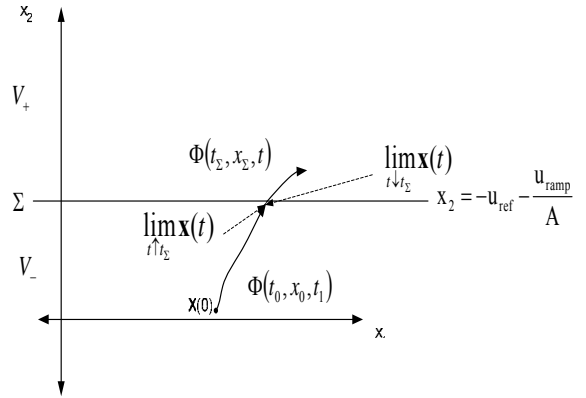


Fig. 4 Derivation of the saltation matrix

In which  $\mathbf{S}$  (called saltation matrix) is the matrix that defines the solution on the hypersurface at  $t = t_\Sigma$  given by [6]:

$$\mathbf{S} = \mathbf{I} + \frac{\left( \lim_{t \downarrow t_\Sigma} (\mathbf{f}_+(\mathbf{x}, t)) - \lim_{t \uparrow t_\Sigma} (\mathbf{f}_-(\mathbf{x}, t)) \right) \mathbf{n}^T}{\mathbf{n}^T \lim_{t \uparrow t_\Sigma} (\mathbf{f}_-(\mathbf{x}, t)) + \frac{\partial h}{\partial t}(\mathbf{x}, t_\Sigma)} \quad (22)$$

$$\text{where } \lim_{t \uparrow t_\Sigma} (\mathbf{f}_-(\mathbf{x}, t)) = \begin{bmatrix} \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} \\ \frac{v_{in} - x_1(t_\Sigma)}{L} \end{bmatrix},$$

$$\lim_{t \downarrow t_\Sigma} (\mathbf{f}_+(\mathbf{x}, t)) = \begin{bmatrix} \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} \\ -\frac{x_1(t_\Sigma)}{L} \end{bmatrix} \text{ and}$$

$$\lim_{t \downarrow t_\Sigma} (\mathbf{f}_+(\mathbf{x}, t)) - \lim_{t \uparrow t_\Sigma} (\mathbf{f}_-(\mathbf{x}, t)) =$$

$$\begin{bmatrix} \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} \\ \frac{v_{in} - x_1(t_\Sigma)}{L} \end{bmatrix} - \begin{bmatrix} \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} \\ -\frac{x_1(t_\Sigma)}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{v_{in}}{L} \end{bmatrix}$$

The numerator of (22) can be calculated to

$$\left( \lim_{t \downarrow t_\Sigma} (\mathbf{f}_+(\mathbf{x}, t)) - \lim_{t \uparrow t_\Sigma} (\mathbf{f}_-(\mathbf{x}, t)) \right) \mathbf{n}^T = \begin{bmatrix} 0 \\ \frac{v_{in}}{L} \end{bmatrix} [1 \ 0] = \begin{bmatrix} 0 & 0 \\ \frac{v_{in}}{L} & 0 \end{bmatrix}.$$

The first part of the denominator of (22) can be expressed to

$$n^T \lim_{t \rightarrow t_\Sigma} (\mathbf{f}_-(\mathbf{x}, t)) = [1 \quad 0] \begin{bmatrix} \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} \\ \frac{v_{in} - x_1(t_\Sigma)}{L} \end{bmatrix} = \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C}$$

and the second part of the denominator of (22) can be determined to:

$$\frac{\partial h}{\partial t}(\mathbf{x}, t_\Sigma) = \frac{\partial \left( x_1 - U_{ref} - \frac{V_L + (V_U - V_L)t}{AT} \right)}{\partial t} = -\frac{(V_U - V_L)}{AT}$$

The saltation matrix  $\mathbf{S}$  becomes now:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ \frac{v_{in}}{L} & 1 \\ \frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} - \frac{(V_U - V_L)}{AT} & 1 \end{bmatrix}$$

$$\text{And } \Phi(t_0, x_0, t) = \Phi(t_\Sigma, x_\Sigma, t) \mathbf{S} \lim_{t_1 \rightarrow t_\Sigma} \Phi(t_0, x_0, t_1) \quad (23)$$

So if  $\mathbf{S}$  is known it is possible to find the eigenvalues of  $\Phi(t_0, x_0, t)$  which for period 1 must have amplitude less than 1.

## V. CONCLUSION

The effect of the discontinuity of the buck converter's vector field on the fundamental matrices, which are defined in the smooth areas separated by the switching hypersurface, has been studied by applying Filippov inclusions to explain the behavior of the system on that hypersurface. A mathematical description is presented of the state solutions before, after and, for the first time, during the crossing of the hypersurface, as defined by the switching discontinuities of the converter. The work presented here can be used to study the stability and the bifurcation phenomena of the buck converter either by using Lyapunov functions (presented in [7] for discontinuous systems) or by using the Floquet multipliers of the monodromy matrix.

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