A state-space approach for analysing the bullwhip effect in supply chains

Christos Papanagnou and George Halikias
Control Engineering Research Centre
School of Engineering and Mathematical Sciences
City University,
London EC1V 0HB, UK
Email: {c.papanagnou, g.halikias}@city.ac.uk

***Abstract*— The bullwhip effect is a well known instability phenomenon in supply chains, related to volatility amplification of demand profiles in the upper nodes of the chain. We propose a novel state-space approach for analysing a simple series supply chain model with an arbitrary number of nodes. In addition, we develop techniques for calculating explicitly the associated covariance matrix in parametric form, under white-noise demand profile assumptions. This allows us to analyze the effect of a parameter in the studied inventory policies on the bullwhip effect for chains with an arbitrary number of nodes.**

Index terms- **Bullwhip effect, covariance matrix, state-space analysis, supply chain.**

I. INTRODUCTION

The bullwhip effect is a well known instability phenomenon in supply chains, related to increased volatility in demand profiles in the upstream nodes of the chain. This may limit significantly the smooth operation of the chain and result in high costs arising due to its implications on production planning, high levels of inventory costs, poor customer service, etc. The bullwhip effect has been analyzed extensively in recent literature, and many contributing factors for this phenomenon have been identified [7], [8], [5], [6]. These include poor co-ordination, aggressive stock replenishment/demand forecasting policies and uncertain lead times in the chain.

In this work we develop a simple stochastic multi-node supply chain state-space model and analyze its properties in the steady-state, under white noise demand-profiles of the end-customers. Although a white-noise demand profile is clearly unrealistic for real supply chains (as it ignores, for example, trends, seasonal variations or more complex patterns) it offers the advantage of simplicity and can be easily extended to more complex situations, e.g, ARMA models. The model is sufficient for the purposes of this work, which is the analysis of the effect of inventory policies on the bullwhip effect, rather than, e.g, demand forecasting.

In this work we have opted for a state-space modelling approach rather than the more traditional transfer-function based technique. In a certain sense, state-space and transfer function approaches are equivalent for discrete LTI systems. For example, if a transfer-function technique is followed, the covariance functions of the output variables of the system can be obtained by taking the inverse (two-sided) Z-transform of the spectral density $\Phi(z) = \sigma^2 G(z) G(z^{-1})$, where σ^2 is the variance of the white-noise input and $G(z)$ is the system's transfer function. However, the state-space approach taken in this work is more direct for our purposes and offers the following advantages: (a) State-space methods can be extended to time-varying and non-linear systems, (b) State-space techniques can be used to calculate covariance functions not only of the output variables but also of all internal variables of the system, even "non-minimal" ones, and (c) State-space techniques are more suitable for the recursive updating of the covariance function (obtained by solving a Lyapunov equation) of structured multi-node systems of the type used in this work.

The structure of the paper is as follows. In section 2, we develop a state-space model of a series multi-node chain under simple but realistic assumptions, by adapting the model presented in [4]. We modify this model in two ways: First, our approach is entirely state-space based and does not involve z-transforms. Secondly, we add a stochastic element to the model by assuming that the driving signal is a stochastic process modelling end-customer demand. In this framework, the bullwhip effect is quantified as the variance of the ordering signals, as they propagate upstream in the chain. In section 3, we develop effective computational methods for calculating the covariance matrix of the model in parametric form (for models with an arbitrary number of nodes). This leads in section 4 to the effective characterisation of the bullwhip effect in terms of stock-replenishment policies, assumed to be fully decentralised. Finally, the main conclusions of the work and a few suggestions for further research appear in section 5.

II. THE SUPPLY-CHAIN MODEL

We consider a simple series multi-stage supply chain as shown in Fig.1. There are n individual stages between generic Customer and Manufacturer and we denote as i the intermediate supplier index ($i \geq 1$). Fig.1 also depicts the flow of goods and information(orders) within the supply chain. Let

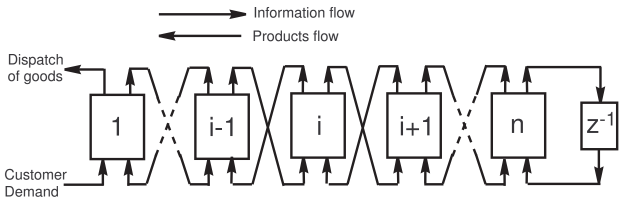

Fig. 1. Series supply chain with n stages

$I_i(t)$ denote the inventory level of node i at time t. We let also $Y_{i,i-1}(t)$ indicate the amount of goods to be delivered to node $i-1$ by the upstream node i at time instant t. We also introduce a time delay L, which is the lead time needed for the goods to be dispatched to the downstream node (i.e, the goods are delivered at time $t+L$). The model is based on [4], from where additional details can be obtained.

Balancing the inventory $I_i(t)$ of node i at time step t gives:

$$I_i(t) = I_i(t-1) + Y_{i+1,i}(t-L) - Y_{i,i-1}(t) \qquad (1)$$

where $I_i(t-1)$ is the inventory level at node i at time step $t-1$ and $Y_{i,i+1}(t-L)$ represents the products dispatched by the upstream node $i+1$ to node i, which is assumed to arrive with a delay of L time steps. Although inventory level is a key variable in supply chain operation, each node i can better monitor the changes in inventory level at time t by using inventory position, $IP_i(t)$, which is given by:

$$IP_i(t) = IP_i(t-1) + Y_{i+1,i}(t) - Y_{i,i-1}(t) \qquad (2)$$

We consider the supply chain network as a decentralised control system - where there is no global moderator and decisions are taken locally at each node, e.g, corresponding to managers aiming to hold their stocks at certain levels or to meet expected future demand, following a series of rules which are known as inventory policies. Hence, the amount of orders placed at the upstream level must satisfy certain criteria such as minimising holding costs, avoiding shortages and maximising profits. Note that each node is the chain is assumed to be autonomous, i.e., we do not consider co-operation between managers through information sharing [1]. The resulting decentralised structure of the system is one of the main contributory factors of the bullwhip effect - in a sense it is analogous to the “string oscillations” observed in automated highway systems due to the lack of proper “preview information”. Whereas co-operation between SC managers (e.g., by sharing information) would certainly help to alleviate the bullwhip effect, unfortunately most enterprises regard these data as proprietary and are reluctant to share them. Techniques for forcing co-operation between SC players are not considered in this work.

We denote by $O_{i,i+1}(t)$ the amount of orders placed by node i to node $i+1$, given by:

$$O_{i,i+1}(t) = k_i(SP_i - IP_i(t)) \qquad (3)$$

where SP_i represents a target set-point (assumed constant) and k_i is the corresponding inventory replenishment gain factor. Standing orders of node i at time step t evolve according to the difference equation:

$$O_i^\star(t) = O_{i-1,i}(t) + O_i^\star(t-1) - Y_{i,i-1}(t) \qquad (4)$$

For the purposes of further analysis it is assumed that there is always enough stock at each node to meet the demand, so that $Y_{i,i-1}(t) = O_i^\star(t-1)$. This implies that the amount of goods dispatched to node i from the downstream stage $i-1$ at time t is the amount of standing orders of node i at time $t-1$. This is essentially a linearisation assumption also made in [4] which simplifies the subsequent analysis. In addition, since the covariance analysis of the following section does not depend on SP_i, we set $SP_i = 0$ for simplicity.

We now consider the series supply chain model depicted in Fig.1. Each stage i has two inputs w_{il} and w_{ir} and two outputs z_{il} and z_{ir} (left and right respectively). It can be inferred from the nature of the figure's interconnections that $w_{il} = z_{il}$ and $w_{ir} = z_{i+1,l}$. The terminal node Φ representing the Manufacturer is assumed to be a simple time delay. Thus the manufacturer always delivers the order he receives with a delay of one time step. The model equations for each separate node can be expressed in state space form as:

$$x_i(t+1) = A_i x_i(t) + \begin{pmatrix} B_{li} & B_{ri} \end{pmatrix} \begin{pmatrix} w_{il} \\ w_{ir} \end{pmatrix}$$

and

$$\begin{pmatrix} z_{il} \\ z_{ir} \end{pmatrix} = \begin{pmatrix} C_{il} \\ C_{ir} \end{pmatrix} x_i(t) + \begin{pmatrix} D_{ill} & D_{ilr} \\ D_{irl} & D_{irr} \end{pmatrix} \begin{pmatrix} w_{il} \\ w_{ir} \end{pmatrix}$$

The equivalent state-space model of the manufacturer is:

$$x_\phi(t+1) = A_\phi x_\phi(t) + B_\phi z_{i+1,r}$$

and

$$w_{i+1,r} = C_\Phi x_\phi(t)$$

Due to our previous assumption we have $A_\phi = 0$ and $B_\phi = C_\phi = 1$. The state space form of the node i given above can be written in more concrete form as:

$$x_i(t+1) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x_i(t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_i(t) \qquad (5)$$

and

$$y_i(t) = \begin{pmatrix} 0 & 1 \\ -k_i & k_i \end{pmatrix} x_i(t) + \begin{pmatrix} 0 & 0 \\ 0 & -k_i \end{pmatrix} u_i(t) \qquad (6)$$

where $y_i(t)$ and $u_i(t)$ are the (two-dimensional) vector outputs and inputs of node i, respectively.

Considering next a supply chain consisting of four stages (e.g, Manufacturer, Distributor, Intermediate Supplier and Retailer) we can easily obtain the overall state space model by aggregating the models of all nodes. In our previous notation this corresponds to setting $i = 2$.

The four-stage supply chain can be described by the following equations:

$$
\begin{aligned}
x_1(t+1) &= A_1 x_i(t) + B_{1r} C_{2l} x_2(t) + B_{11} u(t) \\
x_2(t+1) &= B_{2l} C_{1r} x_1(t) + (A_2 + B_{2l} D_{1rr} C_{2l}) x_2(t) \\
&\quad + B_{2r} C_{3l} x_3(t) \\
x_3(t+1) &= B_{3l} C_{2r} x_2(t) + (A_3 + B_{3l} D_{2rr} C_{3l}) x_3(t) \\
x_\phi(t+1) &= C_{3r} x_3(t) + D_{3rr} x_\phi(t)
\end{aligned}
$$

which can be assembled in matrix form to give the overall model of the chain. Note that here $u(t)$ represents customer's demand.

As mentioned previously, the inventory level I_i and the amount of goods $Y_{i,i-1}$ dispatched by node i to its downstream stage are both important variables for decision making. By making these decisions, managers can manipulate and control the entire supply chain system. By choosing IP_i and $Y_{i,i-1}$ as state space variables, all other variables of the node can be easily calculated.

The state space model given by equations (5) and (6) can be written as:

$$
\begin{aligned}
\begin{pmatrix} IP_i(t) \\ Y_{i,i-1}(t+1) \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} IP_i(t-1) \\ Y_{i,i-1}(t) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} O_{i-1,i}(t) \\ Y_{i+1,i}(t) \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{pmatrix} Y_{i,i-1}(t) \\ O_{i,i+1}(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -k_i & k_i \end{pmatrix} \begin{pmatrix} IP_i(t-1) \\ Y_{i,i-1}(t) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & -k_i \end{pmatrix} \begin{pmatrix} O_{i-1,i}(t) \\ Y_{i,i+1}(t) \end{pmatrix} \\
&+ \begin{pmatrix} 0 \\ k_i \end{pmatrix} SP_i(t)
\end{aligned}
$$

Note that the assumed inventory replenishment policy is continuous (rather than periodic). We also not consider variations in the setpoint levels, i.e., $SP_i(t)$ are assumed to be constant.

III. Computation of model's covariance matrix

In this section we outline a method for calculating the covariance matrix of the state-vector $x(t)$ of the overall model developed in the previous section using symbolic computations. In our application, symbolic computations are essential, since we wish to obtain the solution as a function of the gain parameters $\{k_i\}$, which will allow further investigation of the "bullwhip" effect using our model. We first outline a general solution method based on Kronecker matrix products and vector operations [3]; subsequently, the special structure of the state-space model is exploited to derive a simple recursive solution procedure which can be applied to models of arbitrarily high complexity.

Consider the LTI discrete-time state-space model:

$$x(t+1) = Ax(t) + Be(t),\ y(t) = Cx(t) + De(t) \qquad (7)$$

where $\{e(t)\}$ denotes a white vector-noise sequence of unit intensity, representing customer demand, assumed to have been applied as input to the model since the infinite past. Then, assuming that A is asymptotically stable (all eigenvalues of A have modulus less than one), the (steady-state) covariance of the state-vector $x(t)$, $E(x(t)x'(t))$, is given by the (unique, positive semi-definite) solution of the discrete Lyapunov equation [2]:

$$P - APA' - BB' = 0 \qquad (8)$$

Further, $E(yy') = CPC' + DD'$. In our case, A depends linearly on n parameters $k_1, k_2, \ldots, k_n$ which are assumed constant (but possibly unknown). Hence, the solution of (8) is the steady-state covariance of $x(t)$ for all t, for all combinations of $\{k_j\}$ for which A is asymptotically stable. It is shown next that this condition is satisfied if and only if the parameter vector $k = (k_1, k_2, \ldots, k_n)$ lies in the hypercube:

$$\mathcal{K}_n = (0,\ 2)^n := \{k \in \mathcal{R}^n\ :\ 0 < k_i < 2,\ i = 1, 2, \ldots, n\}$$

This agrees with a parallel result in [5].

Lemma 1: Consider the m-stage model (7) depending on m real gain parameters $k = \{k_1, k_2, \ldots, k_m\}$. Then the system is internally stable if and only if $k \in \mathcal{K}_m$. In particular, if $A = A_{2m+1}$ denotes the "A"-matrix of the state-space realisation of the system, then the eigenvalues of A are $\{1-k_1, 1-k_2, \ldots, 1-k_m, 0, \ldots, 0\}$, where the multiplicity of the zero eigenvalue is $m+1$.

Proof: Consider the j-stage state-space model ($j \geq 1$) with corresponding A-matrix given by A_{2j+1}. The proof is by induction on j. For $j = 1$, the eigenvalues of A_3 may be easily calculated as $\{1-k_1, 0, 0\}$. Thus A_3 is asymptotically stable if and only if $-1 < 1-k_1 < 1$ or equivalently if and only if $0 < k_1 < 2$. Next assume that the eigenvalues of A_{2j-1} are given by $\{1-k_1, \ldots, 1-k_{j-1}, 0, \ldots, 0\}$ (with j zero eigenvalues) for $j \geq 2$, so that A_{2j-1} is asymptotically stable if and only if $(k_1, \ldots, k_{j-1}) \in (0,\ 2)^{j-1}$. Introduce the permutation matrix Q_j, resulting from the interchange of the $(2j-1)$-th and $2j$-th rows and columns of the unit matrix I_{2j+1}. Then,

$$Q_j A_{2j+1} Q_j = \begin{pmatrix} A_{2j-1} & 0 & 0 \\ a'_{21} & 1 & 1 \\ a'_{31} & -k_j & -k_j \end{pmatrix}$$

where a'_{21} and a'_{31} are irrelevant for our present purposes. Thus, since the transformation by Q_j leaves the eigenvalues invariant, the spectrum of A_{2j+1} is given as:

$$
\begin{aligned}
\lambda(A_{2j+1}) &= \lambda(A_{2j-1}) \cup \{1-k_j, 0\} \\
&= \{1-k_1, \ldots, 1-k_j, 0, \ldots, 0\}
\end{aligned}
$$

in which the zero eigenvalue has algebraic multiplicity $j+1$, and hence A_{2j+1} is asymptotically stable if $(k_1, \ldots, k_j) \in (0,\ 2)^j$. This completes the inductive argument. In general, A_{2m+1} has m real eigenvalues at $1-k_j$, $j = 1, 2, \ldots, m$, and another $m+1$ eigenvalues at the origin.

Next, let $A \otimes B$ denote the Kronecker product of two matrices A and B; let also $\text{vec}(A)$ be the operation which stacks the elements of a matrix A in a column vector (sweeping along the rows of A). Applying the $\text{vec}(\cdot)$ operation to (8) and using the identity $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$ (see [3]) gives:

$$(I_{n^2} - A \otimes A)\text{vec}(P) = \text{vec}(BB')$$

which may be solved as:

$$\text{vec}(P) = (I_{n^2} - A \otimes A)^{-1}\text{vec}(BB') \tag{9}$$

The following Lemma guarantees that the indicated inverse in (9) exists. The Lemma shows that matrix $I_{n^2} - A \otimes A$ is non-singular for all $k \in \mathcal{K}$. This is important as it ensures that the symbolic inverse of the matrix exists and can be expressed uniquely as a function of the k_i's. Of course, the solution of the equation is a valid covariance matrix of the state $x(t)$ only when $k \in \mathcal{K}$.

Lemma 2: Matrix $I_{n^2} - A \otimes A$ is non-singular for all $k \in \mathcal{K}$. In fact, $I_{n^2} - A \otimes A$ is singular if and only if $(1-k_i)(1-k_j) = 1$ for any two indices i and j such that $i \le i \le m$ and $1 \le j \le m$, where $n = 2m+1$.

Proof: It follows from standard results on eigenvalues of Kronecker products [3] that $A \otimes A$ has eigenvalues $\{\lambda_i(A)\lambda_j(A) : i, j \in \{1, 2, \ldots, n\}\}$. Thus from Lemma 1, the eigenvalues of $A \otimes A$ are the m^2 products $\{(1-k_i)(1-k_j) : i, j \in \{1, 2, \ldots, m\}\}$ and zero (with multiplicity $n^2 - m^2$). Hence the eigenvalues of $I_{n^2} - A \otimes A$ are the m^2 real numbers $1 - (1-k_i)(1-k_j) = k_i + k_j - k_i k_j$ as i and j vary over the set $\{1, 2, \ldots, m\}$, and one (with multiplicity $n^2 - m^2$). Thus $I_{n^2} - A \otimes A$ is singular if and only if $(1-k_i)(1-k_j) = 1$ for some pair (i, j) such that $1 \le i, j \le m$. Note that this matrix is singular if $k_i = 0$ or $k_i = 2$ for some i, and certainly non-singular if all k_i lie in the interval $(0,\ 2)$.

The covariance matrix obtained in (9) essentially involves the solution of a system of n^2 linear equations in the elements of P, which depend parametrically on the k_i's. Since the solution of the Lyapunov equation is symmetric, however, this system of equations is redundant (with $n(n-1)/2$ equations being repeated). The solution can be simplified using the following procedure: For a symmetric matrix P let $\overline{\text{vec}}(P)$ denote $\text{vec}(P)$ with all the entries of P below the main diagonal eliminated. Clearly, if $P \in \mathcal{R}^{n \times n}$, then $\overline{\text{vec}}(P) \in \mathcal{R}^r$, where $r = n(n+1)/2$. Define $W \in \mathcal{R}^{n^2 \times r}$ so that $\text{vec}(P) = W\overline{\text{vec}}(P)$, e.g, for $n = 2$,

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let also $\mathcal{S} \subset \{1, 2, \ldots, n\}$ be the subset of the $n(n-1)/2$ indices of $\text{vec}(P)$ which are eliminated when constructing $\overline{\text{vec}}(P)$. Then equation (9) may be written as:

$$V(I_{n^2} - A \otimes A)W\overline{\text{vec}}(P) = V\text{vec}(BB') \tag{10}$$

where $V \in \mathcal{R}^{r \times n^2}$ denotes the unit matrix with all rows corresponding to indices in $\mathcal{S}$ eliminated. Clearly, multiplication from the right by matrix V in (10) eliminates the $n(n-1)/2$ redundant equations. Further we have:

Lemma 3: Matrix $V(I_{n^2} - A \otimes A)W$ is non-singular for all $k \in \mathcal{K}$.

Proof: Follows immediately since $I_{n^2} - A \otimes A$ is non-singular for all $k \in \mathcal{K}$, while V and W have full row rank and column rank, respectively.

Thus equation (10) has the unique solution

$$p = \overline{\text{vec}}(P) = [V(I_{n^2} - A \otimes A)W]^{-1}V\text{vec}(BB')$$

from which P can be recovered as $P = \overline{\text{vec}}^{-1}(p)$.

Example: Using the two methods described in the earlier part of this section the covariance matrices corresponding to the one and two stage model were obtained using the symbolic Matlab toolbox as:

$$P_3 = \begin{pmatrix} -\frac{1}{k_1(k_1-2)} & 0 & \frac{1}{k_1-2} \\ 0 & 1 & 0 \\ \frac{1}{k_1-2} & 0 & -\frac{k_1}{k_1-2} \end{pmatrix}$$

and P_5 is given by:

$$\begin{pmatrix} -\frac{1}{k_1(k_1-2)} & 0 & -\frac{(k_1-1}{(k_1-2)k} & \frac{1}{k_1-2} & \frac{(k_1-1)k_2}{(k_1-2)k} \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{k_1-1}{(k_1-2)k} & 0 & -\frac{k+2}{k_1k_2(k_1-2)(k_2-1)k} & \frac{(k_1-1)k_1}{(k_1-2)k} & \frac{(k+2)k_1}{(k_1-2)(k_2-2)k} \\ \frac{1}{k_1-2} & 0 & \frac{(k_1-1)k_1}{(k_1-2)k} & -\frac{k_1}{k_1-2} & -\frac{(k_1-1)k_1k_2}{(k_1-2)k} \\ \frac{(k_1-1)k_2}{(k_1-2)k} & 0 & \frac{(k+2)k_1}{(k_1-2)(k_2-2)k} & -\frac{(k_1-1)k_1k_2}{(k_1-2)k} & -\frac{(k+2)k_1k_2}{(k_1-2)(k_2-2)k} \end{pmatrix}$$

respectively, where $k = k_1k_2 - k_2 - k_1$.

A still better method for calculating the covariance matrix of the state-vector is to use the special structure of the state-space model, which leads to a simple recursive updating algorithm. This is outlined in the following result:

Lemma 4: Let (A_{2j+1}, B_{2j+1}) denote the j-stage state-space model, depending on the j parameters $\{k_1, k_2, \ldots, k_j\}$ where $j \ge 1$. Then:

1) There is a state-space transformation defined by a permutation matrix Q_j, such that

$$Q_jA_{2j+1}Q_j := A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

and

$$Q_jB_{2j+1} = B_{2j+1} := B$$

in which: (i) $A_{11} = A_{2j-1}$ and, (ii) A_{21} and A_{22} have rank one, and (iii) B is of the form $[B_1' \; 0_{2j-1}]'$.

2) The Lyapunov equation $P - APA' - BB' = 0$ has a unique symmetric positive-semidefinite solution P for all $(k_1, k_2, \ldots, k_j) \in (0,2)^j$. Let P be partitioned conformally with A, i.e,

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix}$$

where $P_{11}' = P_{11} \in \mathcal{R}^{(2j-1)\times(2j-1)}$, $P_{12} \in \mathcal{R}^{(2j-1)\times 2}$ and $P_{22}' = P_{22} \in \mathcal{R}^{2\times 2}$. Then $P_{11} = P_{2j-1}$ where P_{2j-1} is the covariance matrix of the $(j-1)$-th stage model, i.e, the unique symmetric solution of the discrete Lyapunov equation:

$$P_{2j-1} - A_{2j-1}P_{2j-1}A_{2j-1}' - B_{2j-1}B_{2j-1}' = 0$$

Further, P_{12} and P_{22} have rank at most one and may be obtained from the unique solutions of the linear equations:

$$P_{12} - A_{11}P_{12}A_{22}' = A_{11}P_{11}A_{21}'$$

and

$$\begin{aligned} P_{22} - A_{22}P_{22}A_{22}' &= A_{21}P_{11}A_{21}' + A_{22}P_{12}'A_{21}' \\ &+ A_{21}P_{12}A_{22}' \end{aligned}$$

respectively.

3) If $(k_1, k_2, \ldots, k_j) \in (0,2)^j$, the Lyapunov equation:

$$P_{2j+1} - A_{2j+1}P_{2j+1}A_{2j+1}' - B_{2j+1}B_{2j+1}' = 0$$

has a unique symmetric positive semi-definite solution given by:

$$P_{2j+1} = Q_j \begin{pmatrix} P_{2j-1} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix} Q_j$$

Remark: The Lemma shows that the covariance matrix of the j-th stage model may be obtained recursively from the solution of the $(j-1)$-th stage model by solving two linear equations of order $2(2j-1)$ and 4, respectively (in fact of order $2j-1$ and 2, taking into account that P_{12} and P_{22} have both rank at most one). This can be achieved by the vectorization approach outlined earlier. In any case, the bulk of the computation involving the solution of a $(2j-1)\times(2j-1)$ matrix equation is completely avoided. After P has been assembled from P_{2j-1}, P_{12} and P_{22}, P_{2j+1} may be obtained by reversing the permutation through matrix Q_j.

Proof: The decomposition of $Q_jA_{2j+1}Q_j$ and the fact that $A_{11} = A_{2j-1}$ follows directly from Lemma 1. Further note that

$$A_{21} = \begin{pmatrix} O_{2j-2} & -1 \\ O_{2j-2} & k_2 \end{pmatrix} \quad \text{and} \quad A_{22} = \begin{pmatrix} 1 & 1 \\ -k_j & -k_j \end{pmatrix}$$

so that both A_{21} and A_{22} have rank one. The fact that $Q_jB_{2j+1} = B_{2j+1}$ also follows immediately since the only non-zero element of B_{2j+1} is the second.

Since A is asymptotically stable for $(k_1, k_2, \ldots, k_j) \in (0,2)^j$, the discrete-time Lyapunov equation $P - APA' - BB' = 0$ has a unique symmetric positive-semidefinite solution [2]. Using the indicated partitioning, this may be written as:

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix}\begin{pmatrix} A_{11}' & A_{21}' \\ 0 & A_{22}' \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix} + \begin{pmatrix} B_{2j-1}B_{2j-1}' & 0 \\ 0 & 0 \end{pmatrix}$$

which is equivalent to the three matrix equations:

$$\begin{aligned} P_{11} - A_{11}P_{11}A_{11}' &= B_3B_3' \\ P_{12} - A_{11}P_{12}A_{22}' &= A_{11}P_{11}A_{21}' \end{aligned}$$

and

$$P_{22} - A_{22}P_{22}A_{22}' = A_{21}P_{11}A_{21}' + A_{22}P_{12}'A_{21}' + A_{21}P_{12}A_{22}'$$

Note that the first of these is a discrete Lyapunov equation; since A_{11} is asymptotically stable the solution of this equation is unique, and hence $P_{11} = P_{2j-1}$. Moreover, since A_{11} and A_{22} are both asymptotically stable, the solutions of the second and third equations are also unique [2] and P_{22} is positive semidefinite. To show that P_{12} and P_{22} have both rank at most one, note that the second and third equations may be written as:

$$P_{12} = A_{11}\begin{pmatrix} P_{11} & P_{12} \end{pmatrix}\begin{pmatrix} A_{21}' \\ A_{22}' \end{pmatrix}$$

and

$$P_{22} = \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} P \begin{pmatrix} A_{21}' \\ A_{22}' \end{pmatrix}$$

Now,

$$\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} O_{2j-2} & -1 & 1 & 1 \\ O_{2j-2} & k_j & -k_j & -k_j \end{pmatrix}$$

has rank one, and hence P_{12} and P_{22} have rank at most one. Finally, note that if $(k_1, k_2, \ldots, k_j) \in (0,2)^j$, A_{2j+1} is asymptotically stable and hence the Lyapunov equation:

$$P_{2j+1} - A_{2j+1}P_{2j+1}A_{2j+1}' - B_{2j+1}B_{2j+1}' = 0$$

has a unique solution. The recursive updating formula

$$P_{2j+1} = Q_j \begin{pmatrix} P_{2j-1} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix} Q_j$$

now follows on noting that under the state-space transformation $A_{2j+1} \rightarrow Q_jA_{2j+1}Q_j$, $B_{2j+1} \rightarrow Q_jB_{2j+1} = B_{2j+1}$, the solution of the Lyapunov equation corresponding to (A_{2j+1}, B_{2j+1}) transforms as $P_{2j+1} \rightarrow Q_jP_{2j+1}Q_j$.

IV. Characterisation of Bullwhip effect

The covariance analysis carried out in the previous section allows us to analyze the effect of the inventory replenishment policies on the bullwhip effect. Recall that end-customer demand has been modelled as a white-noise sequence. Hence, the variance of the demand signal at any node of the chain may be calculated easily from the covariance matrix. Consider as

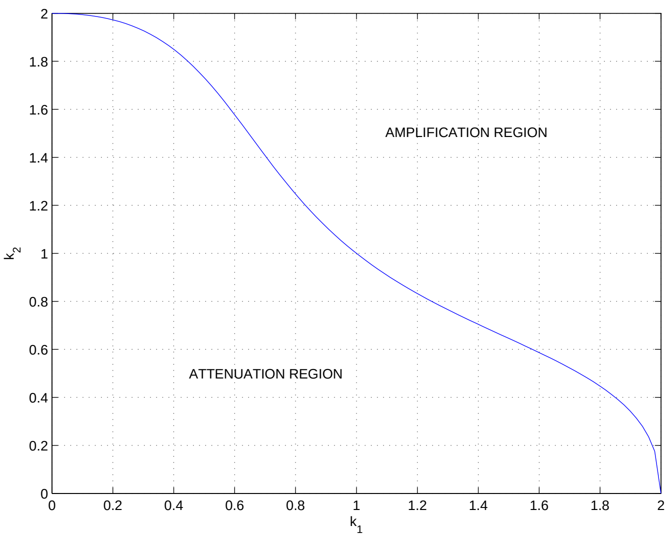

Fig. 2. Boundary between demand amplification and attenuation regions

an example a two-stage supply chain model. The orders placed by the second node (on the manufacturer) correspond to signal z_{2r} and we can write:

$$z_{2r}(t) = C_{2r}x_2(t) + D_{22r}w_{2r}(t) = C_{2r}x_2(t) - k_2 x_\Phi(t)$$

or

$$z_{2r}(t) = Cx(t)$$

where

$$C = \begin{pmatrix} 0 & 0 & -k_2 & k_2 & -k_2 \end{pmatrix}$$

and

$$x'(t) = \begin{pmatrix} x_1' & x_2' & x_\Phi \end{pmatrix}$$

Thus the demand amplification factor can be obtained from the variance of z_{2r}, σ^2 which is given as:

$$\sigma^2 = E(z_{2r}^2) = CP_5C' = \frac{k_1k_2(2 + k_1k_2 - k_1 - k_2)}{(2-k_1)(2-k_2)(k_1k_2 - k_1 - k_2)}$$

To find the regions in the (k_1, k_2) plane where demand amplification and demand attenuation occurs, this expression was set to one, and the resulting equation was solved to give k_2 as a function of k_1 (detailed expressions are omitted). The resulting curve is plotted in Fig.2, and indicates the boundary between the demand-amplification and demand-attenuation regions. As expected, aggressive replenishment policies (i.e, large values of k_1 and k_2) reinforce the bullwhip effect. Although extensive simulation results have been obtained illustrating the validity of the formula for the amplification factor, these are not included in the paper due to lack of space, but will be presented and discussed during the Conference presentation.

V. CONCLUSION

A novel state-space model has been presented for analysing the bullwhip effect in a simple multi-stage supply chain model using realistic assumptions. Effective symbolic computation methods have been presented for calculating the covariance matrix of the model in parametric form, under white-noise customer demand profiles, applicable to models with an arbitrary number of nodes. This was used to obtain an efficient characterisation of the bullwhip effect for cascade multinode chains in terms of inventory replenishment policies. Future work will attempt to use the additional information provided by the structure of the covariance matrix to estimating unknown parameters of the system (e.g, future demand profiles, replenishment policies of neighbouring nodes, etc) hopefully leading to an effective decentralised control scheme.

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