The TOR design for the optimal control of linear systems

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***Abstract* - In this paper the theoretical base and practical application of the TOR design for the control of linear systems is presented. This method is based on the synergy of the LQG balancing and Persson-Astrom (PA) methods for the design of optimal controllers. LQG balancing is a base for the order reduction of linear systems, whose processes and measurements are affected by white noise. The Persson-Astrom (PA) method is a base for the tuning of PID controllers. The method of calculation of the LQG controller is presented, in which the Riccati equations and Kalman estimator are used. The superiority of the TOR idea: the simultaneous use and interconnection of PA-PID tuning and LQG method is pointed out, by its comparison to the single use of the PID approach and LQG approach, which are provided in the MATLAB demo for the control of angular velocity of a DC motor exposed to external disturbances (torque Td). The control strength of various methods is presented at the comparative diagram (Fig. 2) developed by programming in the MATLAB environment.**

Index terms – LQG Balancing, model reduction, Persson-Astrom method, TOR, controller, compensator, MATLAB

1. INTRODUCTION

The TOR method is a synergy of the method of tuning PID controllers, introduced by Persson and Astrom [16],[17], and the method of balancing, introduced by Moore [1].

Moore used the idea of balancing as a method for the reduction of the order of linear systems. Ever since, the balancing method has been extended to various directions, such as the balancing of unstable linear systems [2], [3], the balancing of conservative mechanical systems [4] as well as the balancing of nonlinear systems (stable and unstable), explored by Scherpen [5]. The superiority of the application of balancing method for the order reduction of systems has been investigated by Glover [6]. In addition, several modifications of the balancing technique have been proposed with purpose to improve its flexibility and efficiency [7-11]. Among the methods proposed for the order reduction of linear systems, the Hankel approach, the *Linear-Quadratic-Gaussian (LQG)* approach and the H_∞ approach stand out.

This paper's focus is on the LQG balancing and the search of its applications for the optimal control of linear systems.

2. LQG BALANCING

The application of the Hankel method for linear and stable systems [6],[23] has been proved successful in reducing the order of an open-loop system. Although, nobody can know beforehand whether it can achieve a successful approximation of the respective closed-loop system. Thus, the method of LQG balancing, introduced by Jonckheere and Silverman [10],[11], was chosen in this paper.

The LQG balancing is formulated for minimal linear systems, with state equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx + v \end{aligned} \qquad (1)$$

where (A, B, C) are the matrices of the system, x is its state vector, u its input vector, y its output vector ($u \in R^m$, $x \in R^n$ and $y \in R^p$), while d and v are independent white Gaussian noise processes, with covariance functions $I*\delta(t-\tau)$.

The optimal compensator (regulator) for system (1) has to minimize the following cost function

$$J(x_0, u(.)) = E \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[x^T(t)\, C^T C\, x(t) + u^T(t)\, u(t)\right] dt \qquad (2)$$

According to Jonckheere, Silverman [11] and Opdenacker [12], this compensator is determined by the equations

$$\begin{aligned} \dot{z} &= Az + Bu + SC^T(y - Cz) \\ u &= -B^T Pz \end{aligned} \qquad (3)$$

where,
S is the solution of the algebraic Riccati equation

$$AS + SA^T + BB^T - SC^T CS = 0 \qquad (4)$$

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concerning the system filter (Filter Algebraic Riccati Equation-FARE), and
P is the stabilizing solution of the Riccati equation (5) relating to the control of the system (Control Algebraic Riccati Equation-CARE)

$$A^T P + PA + C^T C - PBB^T P = 0 \qquad (5)$$

Theorem 1. [7], [8], [9],[23]. The eigenvalues of P, S are invariant under equivalence transformations. There exists a representation on the state space such that

$$M := P = S = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \qquad (6)$$

with $\mu_1 \geq \mu_2 \geq ...\mu_n \geq 0$. This representation is called LQG balanced representation or LQG balanced form.

If $\mu_k \geq \mu_{k+1}$, then the components $x_1....x_k$ of the state vector are more difficult to be controlled and filtered than the components $x_{k+1}....x_n$. Thus, a model based only on the components $x_1....x_k$, it could conserve possibly the substantial properties of the initial system in a structural closed-loop synthesis.

By considering the state vector of system (1) as partitioned to its first k components and its last n-k components, the system's matrices are partitioned as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \qquad (7)$$

$$x^1 = (x_1,...,x_k)^T, \; x^2 = (x_{k+1},...,x_n)^T,$$

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

where
$\Sigma_1 = diag(\sigma_1,...,\sigma_k)$ and $\Sigma_2 = diag(\sigma_{k+1},...,\sigma_n)$.

The equivalent reduced-order system is then

$$\begin{aligned} \dot{x} &= A_{11}x + B_1 u + B_1 d \\ y &= C_1 x + v \end{aligned} \qquad (8)$$

Theorem 2. [7],[8],[9],[23]. If $\mu_k \geq \mu_{k+1}$, then: (i) the system (A_{11}, B_1, C_1) is minimal, (ii) the reduced-order system (8) is also LQG balanced, and (iii) the optimal compensator of system (8) is the reduced-order compensator of the full rank system (1).

The idea of balancing is based on the input energy level that the state vector of the system is required to reach, as also on the output energy produced by that vector. To define these functions, for system (1) the following minimal system without noise is considered:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad (9)$$

where $u \in R^m$, $x \in R^n$ and $y \in R^p$.

The energy functions, then, corresponding to the past energy $K^-(x_0)$ and the future energy $K^+(x_0)$ of system's (9) state vector x_0 (t=0), are defined by the relations

$$K^-(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\; x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} (\| y(t) \|^2 + \|u(t)\|^2) dt \qquad (10)$$

$$K^+(x_0) = \min_{\substack{u \in L_2(0,\infty) \\ x(\infty)=0,\; x(0)=x_0}} \frac{1}{2} \int_{0}^{\infty} (\| y(t) \|^2 + \|u(t)\|^2) dt \qquad (11)$$

Theorem 3.[13] If S and P are the solutions of the Riccati equations (4) and (5), then the energy equations are

$$K^-(x_0) = \frac{1}{2} x_0^T S^{-1} x_0, \text{ and } K^+(x_0) = \frac{1}{2} x_0^T P x_0 \qquad (12)$$

By taking into account relation (6), we receive then

$$K^-(x_0) = \frac{1}{2} x_0^T M^{-1} x_0 \text{ and } K^+(x_0) = \frac{1}{2} x_0^T M x_0 \qquad (13)$$

where M is diagonal matrix.

From the energy point of view, the significance of a component x_i of the state vector $\bar{x} = (0,...,0,x_i,0,...,0)$ is measured in relation to its respective invariant value μ_i $(i = 1,...n)$. For small values of μ_i, the amount of input energy required by the system to reach its state $\bar{x}$ is large, while the output energy produced by $\bar{x}$ is small. Hence, if $\mu_k >> \mu_{k+1}$, then the components $x_{k+1}...x_n$ of the state vector are less important and can be neglected; thus, the order of the model describing the initial system is reduced [3].

3. DESIGN OF THE LQG REGULATOR

Bounded Output Linear-Quadratic State-Feedback Regulator

Given the linear minimal system (9) (or the equivalent discrete-time system), we can design a state-feedback control u=-Kx, which minimizes the quadratic cost function with bounded output

$$J(u) = \int_0^\infty (y^T Q\, y + u^T R\, u + 2y^T N\, u)\, dt \qquad (14)$$

Linear-Quadratic- Gaussian controller design

LQG control is a modern technique for designing an optimal dynamic regulator. It has been developed for both continuous and discrete systems and permits the consideration of the disturbances in the system and noise as well, during the measurement of its output.

The objective here is to adjust the output y around zero. The system is affected by disturbances w and is controlled by the vector u. In order to produce the control action u, the controller is receiving measurements which include noise: $\bar{y} = y + \upsilon$ (measurement + noise). The equations describing the system are

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ \bar{y} &= Cx + Du + Hw + \upsilon \end{aligned} \qquad (15)$$

where w and υ are both white noises.

The calculation of the LQG regulator requires the calculations: (i) of the optimal state feedback gain K, which is received as the result of the minimization procedure (14), and (ii) of the Kalman estimator $\hat{x}$.

The estimation of the state vector is produced by using a Kalman filter that minimizes the asymptotic covariance $\underbrace{\lim}_{t \to \infty} E((x - \hat{x})(x - \hat{x})^T)$ of the estimated value of the error $x - \hat{x}$. The estimator $\hat{x}$ is given by the equation

$$\frac{d}{dt}\dot{\hat{x}} = A\hat{x} + Bu + L(\bar{y} - C\hat{x} - Du) \qquad (16)$$

where u is the control input vector and $\bar{y}$ the received measurements. The Kalman gain L can be computed from a Riccati equation, where

$$E(ww^T) = Q_n,\ E(\upsilon\upsilon^T) = R_n,\ E(w\upsilon^T) = N_n \quad (17)$$

The calculated, then, control action u = -K $\hat{x}$ remains optimal for the system.

The MATLAB environment offers routines appropriate for the above calculation.

4. Illustrative example in the MATLAB environment

A. Problem definition

1. Consider the DC motor given in the MATLAB demo [21]. By using the Control System Analysis and Design Toolbox of MATLAB, a bounded output LQG regulator to be designed in order the impact of the load disturbances (torque Td) upon the angular velocity ω(t) of the motor to be reduced to the minimum.
2. The TOR (Tsiantis Optimal Regulator) design to be developed through the simultaneous exploitation of the LQG design and of the Persson-Astrom method of tuning PID controllers ([16],[17],[22]). The TOR design to be tested in the MATLAB environment and its performance to be compared to the MATLAB demo, the single PA-PID tuning method, the single LQG method and other methods as well.

B. Mathematical description of the problem

According to the theoretical background [19], [20], [23], the system of D.C. motor can be described by the linear vector equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (18)$$

In matrix form, the system is described as follows:

$$\begin{bmatrix} \dot{\omega} \\ \dot{i}_a \end{bmatrix} = \begin{bmatrix} -\frac{K_f}{J} & \frac{K_m}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \omega \\ i_a \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{J} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} u_a \\ u_d \end{bmatrix} \qquad (19)$$

$$y = [1 \quad 0] \begin{bmatrix} \omega \\ i_a \end{bmatrix} + [0 \quad 0] \begin{bmatrix} u_a \\ u_b \end{bmatrix} \qquad (20)$$

where

x= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ i_a \end{bmatrix}$ the system state vector,

$$\dot{x} = \frac{dx}{dt} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \omega \\ i_a \end{bmatrix} = \begin{bmatrix} \dot{\omega} \\ \dot{i}_a \end{bmatrix}$$

u= $\begin{bmatrix} u_a \\ u_d \end{bmatrix} = \begin{bmatrix} V_a \\ T_d \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ the system input vector,

y, the system output vector (single output), y=ω,
A, the state vector matrix,
B, the input vector matrix,
C, the output vector matrix, and
D=0.

The values of the physical quantities of the motor are:
R=2.0Ohms, L=0.5 Henry, $K_m = K_b = 0.1$,

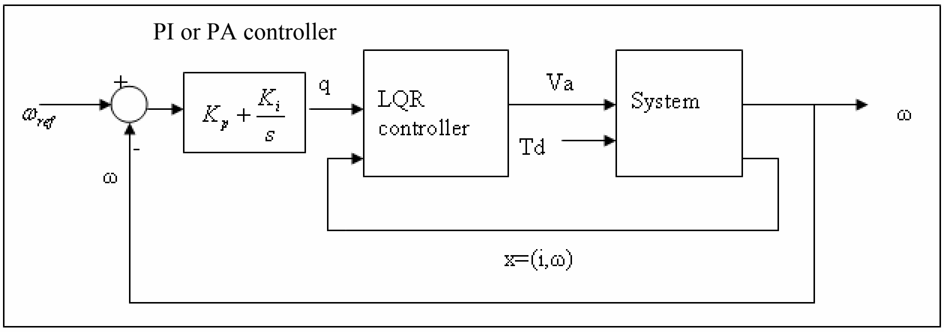

Fig. 1. Construction of the control system and connection of the controller to a tuned PI-control (Persson-Astrom).

$K_f = 0.2$ Nms, J=0.02 $Kg\ m^2\ s^{-2}$.

By replacing the values of symbols in the matrices, the state equations of the D.C. motor are taking then the following form:

$$\begin{bmatrix} \dot{\omega} \\ \dot{i}_a \end{bmatrix} = \begin{bmatrix} -10 & 5 \\ -0.2 & -4 \end{bmatrix} \begin{bmatrix} \omega \\ i_a \end{bmatrix} + \begin{bmatrix} 0 & -50 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_a \\ T_d \end{bmatrix} \quad (21)$$

$$y = \omega \qquad (22)$$

This M.I.S.O. system (Multiple Input Single Output) has two inputs. By partitioning the input matrix B into two parts: the first representing the component u_a of the input vector u, and the second representing the disturbance component T_d of u, we get the following matrices describing the initial system (sys1):

$$A=\begin{bmatrix} -10 & 5 \\ -0.2 & -4 \end{bmatrix},\ B=\begin{bmatrix} 0 & -50 \\ 2 & 0 \end{bmatrix},\ G=\begin{bmatrix} -50 \\ 0 \end{bmatrix},\ C=[1\quad 0],$$

D=0.

C. Output ω(t) *dip to disturbances Td*

The initial system (sys1) is stable and works appropriately without disturbance, that is, when u=[u1,u2]=[1,0]. Although, when an external disturbance u2 =Td (torque=0.1 Kp.m) is applied to the D.C. motor for a short period of time (5<t<10 sec), then a dip in the angular velocity ω(t) of the motor is noticed and, even, the reverse of its motion. This behavior creates the need for the control of the load disturbances, through the connection of motor to the appropriate controller (compensator). The searching of methods for the design of appropriate controller is examined below.

D. Proportional – Integral Control (PI)

Bu using proportional control, we can bring the input variable u_a at the desired level of the output ω_{ref}. To achieve this objective, we use the inverse static gain K_{ff} of the system's transfer function (sys1), which corresponds to input 1:

$$dcgain(H_1(s)) = |H_1(0)| = \left| \frac{10}{s^2 + 14s + 41} \right|_{s=0} = 0.2439$$

$$K_{ff} = 1/\, dcgain(H_1(s)) = 4.1$$

By using K_{ff} for increasing solely the magnitude of input 1 (u_a) in sys1, we define the open system sys2. The output of sys2 shows some improvement of ω in relation to the effect of Td; nevertheless, the dip of ω(t) has not been eliminated. By choosing a larger or smaller value of K_{ff} than the previous one, the output increases or decreases at the same ratio, without improving the dip of ω(t) and increasing only the error (its distance from the desired value ω_{ref}).

To confront this problem, we use the integral control $h(s) = \frac{K}{s}$, with unity output feedback. If the value of the constant K is smaller of K_{ff}, then the output ω(t) deviates from its desired value ω_{ref}, while if it is larger of K_{ff}, then ω(t) oscillates. We choose, thus, the integrator $h_1(s) = \frac{5}{s}$ and we connect it to the input 1(u_a). Then, sys3 is formed and its output response (I-control) appears on Fig. 2.

E. *Synthesis of LQG and PI control – MATLAB proposal*

If the previous PI control is connected appropriately to the design of a LQG regulator, then sys4a is formed, which corresponds to the MATLAB demo.

F. *PA Control* (tuned PI-control)

We now develop the PA control [24]. From the Persson-Astrom tables [16],[17],[22], for two-term tuning (PI) and characteristic sensitivity M=1.4, we get

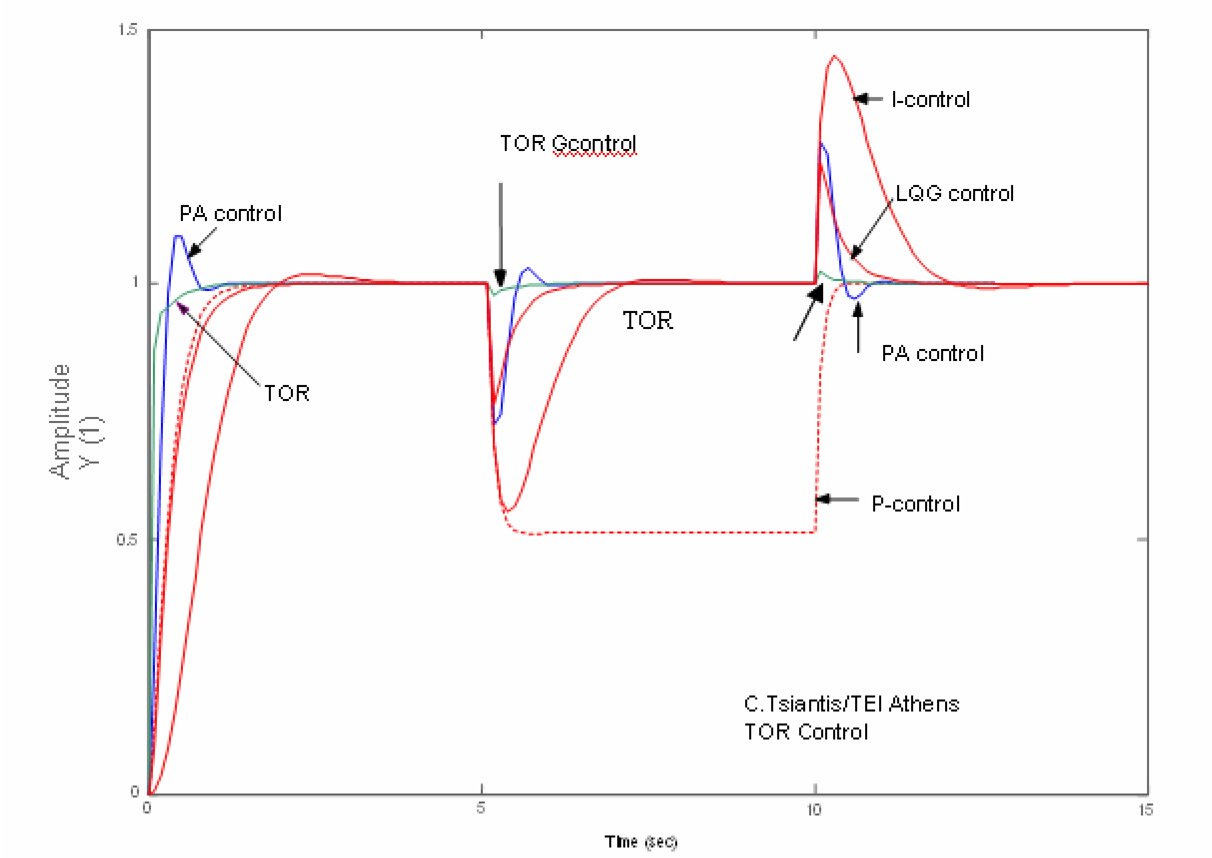

Fig. 2. Comparative diagram of control methods for a DC motor exposed to external load disturbance – Superiority of the TOR approach

$$Ti/T = 0.79\ e^{(-1.4 + 2.4\ \tau)\ \tau} \quad (23)$$

$$K^{*} = \sigma K = 0.29 e^{(-2.7+3.7\tau)\tau} \quad (24)$$

From the step response of sys1, we take the values of σ (=0.03) and τ (=0.111). Then, we calculate the coefficient for the proportional term

K=0.2262/0.03 = **Kp= 7.54**

Given that T=0.36sec and τ=0.111, we compute the integration time Ti=0.697*T=0.25 sec. Thus we take **Ki=30.16**.

The above values provide the tuned Proportional-Integral control law: $h_{PI}(s) = 7.5 + \frac{30.1}{s}$. If this law is applied to the system, it produces the PA-control response, which is illustrated in the comparative diagram (Fig.2).

G. Design specifications of the TOR controller

The design of the TOR dynamic optimal controller [24] is based on the following specifications: (i) the driving control Va(t) of the motor is a linear function of the state vector [ω(t);i(t)] of the motor and its output [ω(t)] as well, (ii) the output [ω(t)] is connected first to the individual PA control and, then, is inserted (as input q(t)) to the LQG regulator, and (iii) the controller satisfies the minimization of the general cost criterion

$$J(u) = \int_{o}^{\infty} (y^T Q\ y + u^T R\ u + 2y^T N\ u)\ dt\,,$$

and specifically

$$J(u) = \int_{o}^{\infty} (20q(t)^2 + \omega(t)^2 + 0.01\, Va(t)^2)\ dt \quad (25).$$

5. Results - Conclusions

The results of this research are resumed on the comparative diagram (Fig.2), which was developed in the MATLAB environment. The following clarifications are accompanied the research procedure: The proportional control (P-control) was computed for the open-loop system, after consideration of the dc gain of the initial system. The integral control (I-control) was computed for the closed-loop system (unity feedback) on the basis of value Ki=5, provided by the MATLAB demo, after examination of the system's root locus. The I-control was improved by using the Persson-Astrom tuned PI controller. The LQG regulator was computed on the basis of the balancing theory and Kalman estimator. The TOR regulator was computed by the simultaneous use of preceding optimal methods: that is, by connecting the system's output to the Persson-Astrom tuned PI controller and by taking as inputs for the LQR regulator: (i) the output of the tuned PI controller, and (ii) the state vector x of the system.

Concluding, the significance of the LQG and PA-tuning methods for the optimal control of linear systems was verified in the diagram (Fig.2). Though, the TOR approach, that is, the ensemble utilization of the LQG controller and Persson-Astrom tuned PI control, it was verified as superior to all the others. Further research, though, is required for the extension and application of the TOR approach.

REFERENCES

[1] B.C. Moore, Principal component analysis in linear systems: Controllability, observability and model reduction. *IEEE Trans. Aut. Contr.*, AC-26 (1981) 17-32.

[2] D.G. Meyer, Fractional balanced reduction-model reduction via a fractional representation, *IEEE Trans. Aut. Contr.* AC-26 (1990), 1341-1345.

[3] R. Ober, D. McFarlane, Balanced canonical forms for minimal systems: A normalized coprime factor approach, *Linear Algebra and its Applications*, (1989), 23-64.

[4] A. J. van der Schaft, J. E. Oeloff, Model reduction of linear conservative mechanical systems, *IEEE Trans. Aut. Control*, AC-35 (1990), 729-733. A. Ball,

[5] J.M.A. Scherpen. *Balancing for Nonlinear Systems*. PhD dissertation. Systems and Control Group. Faculty of Applied Mathematics. University of Twente. 1994. pp.1-163.

[6] K. Glover, All optimal Hankel-norm approximation of linear multivariable systems and their L∞-error bounds, *Int. J. of Contr.*, 39 (1984) 1115-1183.

[7] D.F. Enns, Model reduction with balanced realizations: An error bound and a frequency weighted generalization, *Proc. 23rd Conf.on Dec.and Contr*(1984), 127-132.

[8] L. Pernebo, L.M. Silverman, Model reduction via balanced state space representations, *IEEE Trans. Aut. Contr.*, AC-27 (1982) 382-387.

[9] E.A. Jonckheere, L.M. Silverman, Singular value analysis of deformable systems, *J. of Circ., Syst. and Sign. Proc*, 1 (1982) 447-470.

[10] E.A. Jonckheere, L.M. Silverman, A new set of invariants for linear systems- Applications to reduced order compensator design. *IEEE Tranc. Aut. Contr.*, AS-28 (1983) 953-964.

[11] Ph. Opdenacker ,E.H. Jonckheere, LQG balancing and reduced LQG compensation of symmetric passive systems, *International Journal of Control*, 41, (1985), 73-109.

[12] D. Mustafa, H∞-characteristic values, *Proc. 28th IEEE Conf. on Decomp. and Contr.* (1989) 1483-1487.

[13] A.J. van der Schaft, Nonlinear state space H∞ control theory, στο H.L. Trantelman, J.C. Willems, Eds., *Essays on Control: Perspectives in the theory of its applications*, Birkhauser, Boston (1993) 153-190.

[14] S. Weiland, *Theory of approximation and disturbance attenuation for linear systems*, Doctoral Dissertation, University of Groningen (1991).

[15] S. Weiland, Balancing general quadratic measures on external signals, Progress in Systems and Control Theory: Realization and Modeling in System Theory; Proc. MTNS-89, Vol. 1(1990), 133-140.

[16] P. Persson & K. J. Astrom (1993). "PID control revised", Proc. IFAC 12th World Congress, Sydney.

[17] P. Persson & K. J. Astrom (1992), ''Dominant pole design-a unified vies of PID controller tuning'', Proc. 4th IFAC Symposium on Adaptive Systems in Control and Signal Processing, Grenoble, pp. 127-132.

[18] K.L.Astrom & T. Hagglund (1995). PID Controllers: Theory Design and Tuning, Instrument Society of America.

[19] P. Paraskevopoulos, *Introduction to automatic control*, Volume A:Theory, Athens 2001 (in greek).

[20] S. G. Tzafestas, *Optimal control and pole* placement, Athens - Patra, 1980 (in greek).

[21] Use of the libraries and subroutines of the environment MATLAB 12. Illustrated examples were used for control of our programs.

[22] R. E. King, *Industrial control: Design and tuning of three-term industrial controllers*, Athens, Papasotiriou, 1996, pp84, 92-93,145-147 (in Greek).

[23] C. N. Tsiantis, Balancing and Hankel reduction of continuous time stable systems: Programming and application in the MATLAB environment, *Pan-Hellenic Mechanical-Electrical Engineering Association Magazine*, No.361, May 2003, pp. 68-75.

[24] C. N. Tsiantis. Computation of optimal regulator using LQG balancing and synchronous PID tuning: Programming and application in the MATLAB environment *Pan-Hellenic Mechanical-Electrical Engineering Association Magazine*, under publication (accepted 2003).