Realization problem for positive multivariable continuous-time systems with delays

Tadeusz Kaczorek, *Senior Member, IEEE*
Bialystok Technical University, Poland
kaczorek@isep.pw.edu.pl

***Abstract*— The realization problem for positive, multivariable continuous-time linear systems with delays in state and in control is formulated and solved. Sufficient conditions for the existence of positive realizations of a given proper transfer function are established. A procedure for computation of positive minimal realizations is presented and illustrated by an example.**

I. INTRODUCTION

IN positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [4, 6]. Recent developments in positive systems theory and some new results are given in [7]. Realizations problem of positive linear systems without time-delays has been considered in many papers and books [1, 4, 6].

Explicit solution of equations describing the discrete-time systems with time-delay has been given in [2]. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [3, 11]. The realization problem for positive multivariable discrete-time systems with one time-delay was formulated and solved in [8]. The realization problem for positive continuous-time systems with delays in state has been considered in [9].

The main purpose of this paper is to present a method for computation of positive realizations for positive multivariable continuous-time systems with time-delays in state and in control. Sufficient conditions for the solvability of the realization problem will be established and a procedure for computation of a positive realization of a proper transfer function will be presented.

To the best knowledge of the author the realization problem for positive continuous-time linear systems with delays in state vector and control has not been considered yet.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider the multivariable continuous-time system with h delays in state and q delays in control

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{h} A_i x(t-id) + \sum_{j=0}^{q} B_j u(t-jd) \\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$ are the state, input and output vectors, respectively and $A_i\in\mathbb{R}^{n\times n}$, $i=0,1,\dots,h$ $B_j\in\mathbb{R}^{n\times m}$, $j=0,1,\dots,q$ $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$ and $d>0$ is a delay.

Initial conditions for (1) are given by

$$x_0(t) \text{ for } t\in[-hd,0] \text{ and } u_0(t) \text{ for } t\in[-hq,0] \tag{2}$$

Let $\mathbb{R}_+^{m\times n}$ be the set of $m\times n$ real matrices with nonnegative entries and $\mathbb{R}_+^m=\mathbb{R}_+^{m\times 1}$.

Definition 1. The system (1) is called (internally) positive if for every $x_0(t)\in\mathbb{R}_+^m$, $t\in[-hd, 0]$, $u_0(t)\in\mathbb{R}_+^m$, $t\in[-qh, 0]$ and all inputs $u(t)\in\mathbb{R}_+$, $t\geq 0$ we have $x(t)\in\mathbb{R}_+^m$ and $y(t)\in\mathbb{R}_+$ for $t\geq 0$.

Let M_n be the set of $n\times n$ Metzler matrices i.e. the set of $n\times n$ real matrices with nonnegative off diagonal entries.

Theorem 1. The system (1) is positive if and only if A_0 is a Metzler matrix and matrices A_i, $i=1,\dots,q$, B_j, $j=0,1,\dots,q$, C, D have nonnegative entries, i.e.

$$\begin{aligned}&A_0\in M,\ A_i\in\mathrm{R}_+^{n\times n},\ i=1,..,h,\ B_j\in\mathrm{R}_+^{n\times m},\ j=0,1,\dots,q,\\ &C\in\mathrm{R}_+^{p\times n},\ D\in\mathrm{R}_+^{p\times m}\end{aligned} \tag{3}$$

Proof. To simplify the notation the essence of proof will be shown for $h=q=1$ Using the step method [5 p.49] and defining the vectors

$$\overline{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t+kd) \\ \vdots \\ x_{h+1}(t+kd) \end{bmatrix},\ \overline{u}(t)=\begin{bmatrix} u(t) \\ u(t+d) \\ \vdots \\ u(t+kd) \end{bmatrix}, \tag{4}$$

$$z_0(t)=\begin{bmatrix} A_1x_0(t-d)+B_1u(t-d) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$t\in[0,d]$, (k any positive integer)

and the matrices

$$\overline{A}=\begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ A_1 & A_0 & 0 & \cdots & 0 & 0 \\ 0 & A_1 & A_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_1 & A_0 \end{bmatrix},$$

$$\overline{B}=\begin{bmatrix} B_0 & 0 & 0 & \cdots & 0 & 0 \\ B_1 & B_0 & 0 & \cdots & 0 & 0 \\ 0 & B_1 & B_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & B_1 & B_0 \end{bmatrix}, \tag{5}$$

$$\overline{C}=\begin{bmatrix} C & 0 & \cdots & 0 \end{bmatrix},\ \overline{D}=\begin{bmatrix} D & 0 & \cdots & 0 \end{bmatrix}$$

we may write the equations (1) in the form

$$\begin{aligned} \dot{\overline{x}}(t)&=\overline{A}\overline{x}(t)+\overline{B}\overline{u}(t)+z_0(t) \quad t\in[0,d] \\ y(t)&=\overline{C}\overline{x}(t)+\overline{D}\overline{u}(t) \end{aligned} \tag{6}$$

It is well known [6, 9] that the system (6) is positive if and only if the matrix $\overline{A}$ is a Metzler matrix and the matrices $\overline{B},\overline{C}$ and $\overline{D}$ have nonnegative entries. From the structure of the matrices (5) it follows that the system (1) is positive if and only if (3) holds.

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The transfer function of the system (1) is given by

$$\begin{aligned} T(s,w)&=C[I_ms-A_0-A_1w-\ldots-A_hw^h]^{-1}\times \\ &[B_0+B_1w+\ldots+B_qw^q]+D, \quad w=e^{-hs} \end{aligned} \tag{7}$$

Definition 2. Matrices (3) are called a positive realization of a given transfer function $T(s,w)$ if they satisfy the equality (7). A realization is called minimal if the dimension $n\times n$ of matrices A_i, $i=0,1,\ldots,h$ is minimal among all realizations of $T(s,w)$

The positive realization problem can be stated as follows. Given a proper transfer function $T(s,w)$ Find a positive realization (3) of $T(s,w)$

In this paper sufficient conditions for solvability of the problem will be established and a procedure for computation of a positive minimal realization will be proposed.

III. PROBLEM SOLUTION

The transfer function (7) can be rewritten in the form

$$\begin{aligned} T(s,w)&=\frac{C(\operatorname{Adj}H(s,w))(B_0+B_1w+\cdots+B_qw^q)}{\det H(s,w)}+D= \\ &=\frac{N(s,w)}{d(s,w)}+D \end{aligned} \tag{8}$$

where

$$H(s,w)=[I_ms-A_0-A_1w-\ldots-A_hw^h], \tag{9}$$

$$\begin{aligned} N(s,w)&=C(\operatorname{Adj}H(s,w))(B_0+B_1w+\cdots+B_qw^q), \\ d(s,w)&=\det H(s,w) \end{aligned} \tag{10}$$

From (8) we have

$$D=\lim_{s\to\infty}T(s,w) \tag{11}$$

since $\lim_{s\to\infty}H^{-1}(s,w)=0.$

The strictly proper part of $T(s,w)$ is given by

$$T_{sp}(s,w)=T(s,w)-D=\frac{N(s,w)}{d(s,w)} \tag{12}$$

Therefore, the positive realization problem has been reduced to finding matrices

$$A_0\in M_n,\ A_k\in\mathbb{R}_+^{m\times m},\ k=1,\ldots,q,\ B_j\in\mathbb{R}_+^{m},\ j=1,\ldots,q,\ C\in\mathbb{R}_+^{p\times n} \tag{13}$$

for a given strictly proper transfer function (12).

Lemma 1. If

$$A_0=\begin{bmatrix} 0 & 0 & \ldots & 0 & a_{00} \\ 1 & 0 & \ldots & 0 & a_{01} \\ 0 & 1 & \ldots & 0 & a_{02} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & a_{0n-1} \end{bmatrix},\ A_i=\begin{bmatrix} 0 & 0 & \cdots & 0 & a_{i0} \\ 0 & 0 & \cdots & 0 & a_{i1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{in-1} \end{bmatrix}, \tag{14}$$

$i=1,\ldots,h$

then

$$\begin{aligned} d(s,w)&=\det[I_ns-A_0-A_1w-\cdots-A_hw^h]= \\ &=s^n-d_{n-1}s^{n-1}-d_{n-2}s^{n-2}-\ldots-d_1s+d_0 \end{aligned} \tag{15}$$

where

$$\begin{aligned} d_j&=d_j(w)=a_{h,j}w^h+a_{h-1,j}w^{h-1}+\ldots+a_{1j}w+a_{0j}, \\ j&=0,1,\ldots,n-1 \end{aligned} \tag{16}$$

Proof. Expansion of the determinant with respect to the nth column yields

$$\det[I_n s - A_0 - A_1 w - \ldots - A_h w^h] = \begin{vmatrix} s & 0 & \cdots & 0 & -d_0 \\ -1 & s & \cdots & 0 & -d_1 \\ 0 & -1 & \cdots & 0 & -d_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & s & -d_{n-2} \\ 0 & 0 & \cdots & -1 & s-d_{n-1} \end{vmatrix} =$$

$$s^n - d_{n-1}s^{n-1} - d_{n-2}s^{n-2} - \ldots - d_1 s - d_0$$

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Remark 1. There exist many different matrices $A_0, A_1, \ldots, A_h$ giving the same desired polynomial $d(s,w)$ [8-10].

Remark 2. The matrix A_0 is a Metzler matrix and the matrices $A_1, \ldots, A_h$ have nonnegative entries if and only if the coefficients a_{ij} of the polynomial $d(s,w)$ are nonnegative except $a_{0,n-1}$ which can be arbitrary.

Remark 3. The dimension $n\times n$ of matrices (14) is the smallest possible one for the given $d(s,w)$.

Lemma 2. If the matrices A_i, $i=0,1,\ldots,h$ have the form (14) then the n-th row of the adjoint matrix Adj $H(s,w)$ has the form

$$R_n(s) = [1 \quad s \quad \ldots \quad s^{n-1}] \tag{17}$$

Proof. Taking into account that

$$(\mathrm{Adj}\, H(s,w))\, H(s,w) = I_n d(s,w)$$

it is easy to verify that

$$R_n(s)\, H(s,w) = [0 \quad \ldots \quad 0 \quad 1]\, d(s,w) \tag{18}$$

The strictly proper matrix $T_{sp}(s,w)$ can be always written in the form

$$T_{sp}(s,w) = \begin{bmatrix} \dfrac{N_1(s,w)}{d_1(s,w)} \\ \cdots\cdots\cdots \\ \dfrac{N_p(s,w)}{d_p(s,w)} \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} d_k(s,w) &= s^{n_k} - d_{n_k-1}s^{n_k-1} - \ldots - d_1 s - d_0, \quad k=1,\ldots,p \\ d_i &= d_i(w) = a^i_{h_i i} w^{h_i} + \ldots + a^i_{1i} w + a^i_{0i}, \quad i = 0,1,\ldots,n_k-1 \end{aligned} \tag{20}$$

is the least common denominator of the k-th row of $T_{sp}(s,w)$ and

$$N_k(s,w) = [n_{k1}(s,w), \ldots, n_{km}(s,w)], \quad k=1,\ldots,p \tag{21}$$

$$\begin{aligned} n_{kj}(s,w) &= n_{kj}^{n_k-1} s^{n_k-1} + \ldots + a^1_{1j} w + a^0_{0j}, \quad j = 0,1,\ldots,m \\ n^i_{kj} &= n^{iq}_{kj} w^q + \ldots + n^{i1}_{kj} w + n^{i0}_{kj}, \quad i = 0,1,\ldots,n_k-1 \end{aligned}$$

By Lemma 1 we may associate to the polynomial (20) the matrices

$$A_{k0} = \begin{bmatrix} 0 & 0 & \ldots & 0 & a^k_{00} \\ 1 & 0 & \ldots & 0 & a^k_{01} \\ 0 & 1 & \ldots & 0 & a^k_{02} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & a^k_{0n_k-1} \end{bmatrix}, \quad A_{ki} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a^k_{i0} \\ 0 & 0 & \cdots & 0 & a^k_{i1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & 0 & a^k_{in_k-1} \end{bmatrix}, \ k=1,..,p,\ i=1,\ldots,h_k \tag{22}$$

satisfying the condition

$$\begin{aligned} & d_k(s,w) = \det[I_{n_k} s - A_{k0} - A_{k1} w - \ldots - A_{kh_k} w^{h_k}], \\ & k = 1,..,p \end{aligned} \tag{23}$$

Let

$$\begin{aligned} & A_0 = \text{block diag}\,[A_{10} \ldots A_{p0}] \in \mathrm{R}^{n\times n}, \\ & A_i = \text{block diag}\,[A_{1i} \ldots A_{pi}] \in \mathrm{R}^{n\times n} \quad (n = n_1 + \cdots + n_p) \end{aligned} \tag{24}$$

$$B_k = \begin{bmatrix} b^k_{11} & \cdots & b^k_{1m} \\ \vdots & \ddots & \vdots \\ b^k_{p1} & \cdots & b^k_{pm} \end{bmatrix}, \ b^k_{ij} = \begin{bmatrix} b^{k1}_{ij} \\ \vdots \\ b^{kn_i}_{ij} \end{bmatrix}, \quad k = 0,1,\ldots,q;\ i=1,\ldots,p;\ j=1,..,m \tag{25}$$

$$C = \text{block diag}[c_1 \ldots c_p],\ c_k = [0 \ \ldots \ 0 \ 1] \in \mathrm{R}^{1\times n_k} \ \ k=1,\ldots,p \tag{26}$$

The number of delays q in control is equal to the degree of the polynomial matrix $N(s,w)$ in variable w.

From (8), (17), (22), (24)-(26) we obtain for the j-th column of $T_{sp}(s,w)$

$$T_{sp}^{j}(s,w) = C H^{-1}(s,w)[B_0 + B_1 w + \ldots + B_q w^q]_j = \tag{27}$$

$$= \text{block diag } [c_1 \ldots c_p](\text{block diag}\{[I_{n_1}s - A_{10} - A_{11}w - \ldots$$
$$\ldots - A_{1h_1}w^{h_1}]^{-1}, \ldots, [I_{n_p}s - A_{p0} - A_{p1}w - \ldots - A_{ph_p}w^{h_p}]^{-1}\}) \times$$
$$\begin{bmatrix} b_{1j}^0 + b_{1j}^1 w + \ldots + b_{1j}^q w^q \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_{pj}^0 + b_{pj}^1 w + \ldots + b_{pj}^q w^q \end{bmatrix} =$$
$$= \text{block diag}\left\{ \frac{1}{d_1(s,w)}[1 \quad s \quad \ldots \quad s^{n_1-1}], \ldots \right.$$
$$\left. \ldots, \frac{1}{d_p(s,w)}[1 \quad s \quad \ldots \quad s^{n_p-1}] \right\} \begin{bmatrix} b_{1j}^0 + b_{1j}^1 w + \ldots + b_{1j}^q w^q \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_{pj}^0 + b_{pj}^1 w + \ldots + b_{pj}^q w^q \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{(b_{1j}^{qn_1}w^q + \ldots + b_{1j}^{1n_1}w + b_{1j}^{0n_1})s^{n_1-1} + \ldots + b_{1j}^{q1}w^q + \ldots + b_{1j}^{11}w + b_{1j}^{01}}{d_1(s,w)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{(b_{pj}^{qn_p}w^q + \ldots + b_{pj}^{1n_p}w + b_{pj}^{0n_p})s^{n_p-1} + \ldots + b_{pj}^{q1}w^q + \ldots + b_{pj}^{11}w + b_{pj}^{01}}{d_p(s,w)} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{n_{1j}(s,w)}{d_1(s,w)} \\ \ldots\ldots \\ \dfrac{n_{pj}(s,w)}{d_p(s,w)} \end{bmatrix}, \quad j = 1, \ldots, m$$

and $n_{ij}(s,w)$ are given by (21).

Comparison of the coefficients at the same powers of s and w of the equality (27) yields

$$\begin{gathered} b_{1j}^{01} = n_{1j}^{00}, b_{1j}^{11} = n_{1j}^{01}, \ldots, b_{1j}^{q1} = n_{1j}^{0q}, \ldots, b_{1j}^{0n_1} = n_{1j}^{n_1-1,0}, \\ b_{1j}^{1n_1} = n_{1j}^{n_1-1,1}, \ldots, b_{1j}^{qn_1} = n_{1j}^{n_1-1,q} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{pj}^{01} = n_{pj}^{00}, b_{pj}^{11} = n_{pj}^{01}, \ldots, b_{pj}^{q1} = n_{pj}^{0q}, \ldots, b_{pj}^{0n_1} = n_{pj}^{n_p-1,0}, \\ b_{pj}^{1n_p} = n_{pj}^{n_p-1,1}, \ldots, b_{pj}^{qn_p} = n_{pj}^{n_p-1,q} \end{gathered} \tag{28}$$

for $j = 1, \ldots, m$.

Theorem 2. There exist a positive realization (3) of $T(s,w)$ if

i. $T(\infty) = \lim_{s\to\infty} T(s,w) \in R_+^{p\times m}$ (29)

ii. the coefficients of $d_k(s,w)$ $k=1,\ldots,p$ are nonnegative except a_{0n_k-1}, $k = 1,\ldots,p$, i.e.

$$a_{ij}^k \geq 0, \quad i = 1,\ldots,h_k; \quad j = 0,1,\ldots,n_k - 1, \quad k = 1,\ldots,p \tag{30}$$

iii. the coefficients of $N_j(s,w)$, $j=1,\ldots,m$ are nonnegative, i.e.

$$n_{ij}^k \geq 0 \ \text{ for } i = 1,\ldots,p;\ j = 1,\ldots,m;\ k = 0,1,\ldots,q \tag{31}$$

Proof. The condition (29) implies $D \in \mathbb{R}_+^{p\times m}$ If the conditions (30) are satisfied then the matrices (22) have nonnegative entries except a_{0n_k-1}, $k = 1,\ldots,p$ which can be arbitrary. In this case $A_0 \in M_n$ and $A_i \in \mathbb{R}_+^{n\times n}$, $i=1,\ldots,h$ If additionally the conditions (31) are satisfied then from (28) it follows that $B_k \in \mathbb{R}_+^{n\times m}$, $k=0,1,\ldots,q$. The matrix C of the form (26) is independent of $T(s,w)$ and it has always nonnegative entries.

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Theorem 3. The realization (3) of $T(s,w)$ is minimal if the polynomials $d_1(s,w),\ldots, d_p(s,w)$ are relatively prime (coprime).

Proof. If the polynomials $d_1(s,w),\ldots, d_p(s,w)$ are relatively prime then $d(s,w)=d_1(s,w),\ldots, d_p(s,w)$ and by Remark 3 the matrices (24) have minimal dimensions.

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If the conditions of Theorem 2 are satisfied then a positive minimal realization (3) of $T(s,w)$ can be found by the use of the following procedure.

Procedure

Step 1. Using (11) and (12) find the matrix D and the strictly proper matrix $T_{sp}(s,w)$

Step 2. Knowing the coefficients of $d_k(s,w)$, $k=1,\ldots,p$ find the matrices (14) and (24).

Step 3. Knowing the coefficients of $N_j(s,w)$, $j=1,\ldots,m$ and using (28), (26) find the matrices B_i, $i=0,1,\ldots,q$ and the matrix C.

Example. Using the Procedure find a positive realization (3) of the transfer matrix

$$T(s,w) = \begin{bmatrix} \dfrac{s^2 + (-w^2 + w + 2)s - w^2 + w}{s^2 + (-w^2 + 2)s - (2w^2 + w + 1)} & \dfrac{s^2 + 3s - (2w^2 + 1)}{s^2 + (-w^2 + 2)s - (2w^2 + w + 1)} \\ \dfrac{w^2 + 1}{s - 2w^2 - w + 1} & \dfrac{2s - 2w^2 + 2}{s - 2w^2 - w + 1} \end{bmatrix} \tag{32}$$

It is easy to verify that the assumptions of Theorem 2 are satisfied.

Using the Procedure we obtain

Step 1. From (11) and (12) we have

$$D = \lim_{s\to\infty} T(s,w) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \tag{33}$$

and

$$T_{sp}(s,w) = T(s,w) - D = \\ = \begin{bmatrix} \dfrac{ws + w^2 + 1}{s^2 + (-w^2+2)s - (2w^2 + w + 1)} & \dfrac{(w^2+1)s + w}{s^2 + (-w^2+2)s - (2w^2+w+1)} \\ \dfrac{w^2+1}{s - 2w^2 - w + 1} & \dfrac{2(w^2+w)}{s - 2w^2 - w + 1} \end{bmatrix} \tag{34}$$

Step 2. Taking into account that

$$d_1(s,w) = s^2 + (-w^2+2)s - (2w^2 + w + 1),$$
$$d_2(s,w) = s - 2w^2 - w + 1$$

and using (14), (24) we obtain

$$A_0 = \begin{bmatrix} A_{10} & 0 \\ 0 & A_{20} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & -2 & 0 \\ \hline 0 & 0 & -1 \end{array}\right],\ A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{21} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \\ A_2 = \begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right] \tag{35}$$

Step 3. In this case

$$n_{11}(s,w) = ws + w^2 + 1,\ n_{12}(s,w) = (w^2+1)s + w,$$
$$n_{21}(s,w) = w^2 + 1,\ n_{22}(s,w) = 2(w^2 + w).$$

Using (28) and (26) we obtain

$$B_0 = \begin{bmatrix} b_{11}^{01} & b_{12}^{01} \\ b_{11}^{02} & b_{12}^{02} \\ b_{21}^{01} & b_{22}^{01} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} b_{11}^{11} & b_{12}^{11} \\ b_{11}^{12} & b_{12}^{12} \\ b_{21}^{11} & b_{22}^{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}, \\ B_2 = \begin{bmatrix} b_{11}^{21} & b_{22}^{21} \\ b_{11}^{22} & b_{12}^{22} \\ b_{21}^{21} & b_{22}^{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix},\ \text{and } C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

The desired positive realization of (3) of (32) is given by (33), (35) and (36). The realization is minimal since the polynomials $d_1(s,w)$, $d_2(s,w)$ are relatively prime.

IV. CONCLUDING REMARKS

The realization problem for positive multivariable continuous-time systems with delays in state and in control has been formulated and solved. The special forms (14) and (24) of the matrices A_i, $i=0,1,\dots,h$ have been introduced. Sufficient conditions for the existence of a positive realization (3) of a proper matrix transfer function $T(s,w)$ have been established. The number of delays in control is equal to the degree of the polynomial matrix $N(s,w)$ in the variable w. It has been shown that the realization is minimal if the denominators $d_i(s,w)$, $i=1,\dots,p$ are relatively prime. A procedure for computation of a positive realization (3) of a proper matrix $T(s,w)$ has been presented and illustrated by a numerical example. An extension of the method for singular positive systems with delays is a open problem.

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