Minimal Covering with Stability

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***Abstract*: In this paper we solve two problems. For systems with one input first we find all the *A* modulo *B* invariant subspaces of given dimension that cover a given space and second we find among them those that are stable.**

Index terms. Linear systems, A modulo B invariant subspaces, stability.

I. Introduction

This paper deals with systems described by the equations $\dot{x} = Ax + Bu + Ed$, $y = Cx$

$$\begin{cases} x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, d(t) \in \mathbb{R}^q, y(t) \in \mathbb{R}^r \\ A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, E \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{r\times n} \end{cases} \quad \text{(I.1)}$$

$\mathbb{R}^n$ is the state space, $\mathbb{R}^m$ is the control input space, $\mathbb{R}^q$ is the disturbance input space, $\mathbb{R}^r$ is the output space.

The system is supposed to be controllable.

A subspace $\mathcal{V} \subset \mathbb{R}^n$ is said to be A modulo B invariant if:

$$\exists K \in \mathbb{R}^{m\times n} \text{ with } (A + BK)\mathcal{V} \subset \mathcal{V} \quad \text{(I.2)}$$

In words, an $A \bmod B$ invariant subspace is an eigenspace of a closed loop system by a state feedback.

Suppose that $\dim(\mathcal{V}) = k$ and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the closed loop eigenvalues relative to $\mathcal{V}$ and $v_1, v_2, \cdots, v_k$ the associate eigenvectors. The polynomial $p(\lambda)$ having roots $\lambda_1, \lambda_2, \cdots, \lambda_k$ is the annihilating polynomial of the $A \bmod B$ invariant subspace $\mathcal{V}$. It is convenient to consider sets of roots $\lambda_1, \lambda_2, \cdots, \lambda_k$ closed under complex conjugation in order to be limited to real annihilating polynomials and real $A \bmod B$ invariant subspaces.

The set of all the $A \bmod B$ invariant subspaces of a system is denoted by $\Im(A,B)$. A subspace $\mathcal{V} \in \Im(A,B)$ is said to be stable if its annihilating polynomial is stable. The set of the stable $A \bmod B$ invariant subspaces is denoted by $\Im_S(A,B)$.

The most famous application of the theory of the $A \bmod B$ invariant subspaces [2], is the solution of the disturbance rejection problem. Given $\mathcal{U} \subset \mathbb{R}^n$ there is always one exactly $A \bmod B$ invariant subspace $\mathcal{V}_{\max}$ of maximal dimension contained in $\mathcal{U}$. The disturbance rejection problem has a solution if and only if the biggest $A \bmod B$ invariant subspace $\mathcal{V}_{\max}$ contained in $\operatorname{Ker}(C)$, covers $\operatorname{Im}\text{age}(E)$.

If the problem of the disturbance rejection has not a solution it is useful to restrict the influence of the disturbance. For this purpose one must find the smallest $A \bmod B$ invariant subspace, covering $\operatorname{Im}\text{age}(E)$.

But it is known [1], that there is no a unique $A \bmod B$ invariant subspace of minimal dimension covering a given space. Up to date no algorithm calculating such subspaces is known.

In this paper we present the solution of the single input case. Additionally we discuss questions of stability. The problems solved in this paper are:

Find all the subspaces $\mathcal{V}$ of given dimension with:

$$\begin{array}{l} 1)\mathcal{V} \in \Im(A,B) \text{ and } \mathcal{V} \supseteq \mathcal{E} = \text{Image}(E) \\ 2)\mathcal{V} \in \Im_S(A,B) \text{ and } \mathcal{V} \supseteq \mathcal{E} = \text{Image}(E) \end{array} \quad \text{(I.3)}$$

II. Invariant Subspaces Covering A Given Space

For single input systems, the $A \bmod B$ invariant subspaces are generated by the vector polynomial $x(\lambda) \in \mathbb{R}^n[\lambda]$ for which $\exists\ u(\lambda) \in \mathbb{R}[\lambda]$ with:

$$(\lambda I_n - A)x(\lambda) = Bu(\lambda) \quad \text{(II.1)}$$

We will give a proof for real distinct eigenvalues. Suppose that $t = \left[x(\lambda_1)x(\lambda_2)\cdots x(\lambda_n)\right]$, and $K = \left[u(\lambda_1)u(\lambda_2)\cdots u(\lambda_n)\right]t^{-1}$. Then

$$(A + BK)t = t\,\text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \quad \text{(II.2)}$$

The above equation means that:

$$\text{Span}\left(x(\lambda_1), x(\lambda_2), \cdots, x(\lambda_r)\right) \in \Im(A,B) \quad \text{(II.3)}$$

To include multiple eigenvalues one must consider also derivates of the vector $x(\lambda)$

$$\text{Span}\left(x(\lambda_1), \frac{\partial}{\partial\lambda_1}x(\lambda_1), \cdots, \frac{\partial^q}{\partial\lambda_1^q}x(\lambda_1) \right) \in \Im(A,B) \quad \text{(II.4)}$$

To include complex eigenvalues, $\lambda_k \in \mathbb{C}$

$$\text{Span}\left(\text{Re}\left(x(\lambda_k)\right), \text{Im}\left(x(\lambda_k)\right)\right) \in \Im(A,B) \quad \text{(II.5)}$$

Suppose now that the columns of the matrix t_ε form the controllability canonical basis of the state space. Then

$$(A_\varepsilon, B_\varepsilon, E_\varepsilon) = (t_\varepsilon^{-1}At_\varepsilon, t_\varepsilon^{-1}B, t_\varepsilon^{-1}E_\varepsilon)$$

$$A_\varepsilon = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B_\varepsilon = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, E_\varepsilon = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix}$$

In this case the generator of the $A \bmod B$ invariant subspaces, has the form:

$$x_\varepsilon(\lambda) = \left[1\ \lambda \cdots \lambda^{n-1}\right]^t \quad \text{(II.6)}$$

To develop our results we need the following definitions: Given a vector $\upsilon = [\upsilon_1 \ \cdots \ \upsilon_n]^t \in \mathbb{R}^n$, define the suite of Hankel matrices $\mathcal{H}_k(\upsilon) \in \mathbb{R}^{(n-k)\times(k+1)}$

$\mathcal{H}_0(\upsilon) = \upsilon,$

$$\mathcal{H}_1(\upsilon) = \begin{bmatrix} \upsilon_1 & \upsilon_2 \\ \upsilon_2 & \vdots \\ \vdots & \upsilon_{n-1} \\ \upsilon_{n-1} & \upsilon_n \end{bmatrix}, \mathcal{H}_2(\upsilon) = \begin{bmatrix} \upsilon_1 & \upsilon_2 & \upsilon_3 \\ \upsilon_2 & \upsilon_3 & \vdots \\ \upsilon_3 & \vdots & \upsilon_{n-2} \\ \vdots & \upsilon_{n-2} & \upsilon_{n-1} \\ \upsilon_{n-2} & \upsilon_{n-1} & \upsilon_n \end{bmatrix}, \quad \text{(II.7)}$$

$\mathcal{H}_{n-1}(\upsilon) = [\upsilon_1 \ \cdots \ \upsilon_n]$

Define also the suite of families of polynomials

$$\mathcal{P}(\mathcal{H}_k(\upsilon)) = \{p(s) \mid p(s) = p_0 + p_1 s + \cdots + p_k s^k \in \mathbb{R}[s]\} \quad \text{(II.8)}$$

$$\text{with } \mathcal{H}_k(\upsilon) \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_k \end{bmatrix} = 0_{(n-k)\times 1}$$

An obvious necessary and sufficient condition for a subspace $\mathcal{V} \in \Im(A, B)$ to cover $\mathcal{E}$=Image(E) in the case of distinct real eigenvalues is

$$\text{Span}\left(x(\lambda_1), x(\lambda_2), \cdots, x(\lambda_k)\right) \supset \mathcal{E} \Leftrightarrow$$

$$\exists \psi_1, \psi_2, \cdots, \psi_k \in \mathbb{R} \text{ with } E_\varepsilon = \sum_{\xi=1}^{k} x(\lambda_\xi)\psi_\xi \quad \text{(II.9)}$$

The following theorem is proved for real distinct eigenvalues but it holds for multiple real, complex, and multiple complex eigenvalues as well.

Theorem I

A subspace of the state space $\mathcal{V} \in \Im(A_\varepsilon, B_\varepsilon)$ of dimension k covers the vector E_ε, if and only if its annihilating polynomial $p(s)$ belongs to the family $\mathcal{P}\left(\mathcal{H}_k(E_\varepsilon)\right)$.

Proof

$\mathcal{V} \in \Im(A_\varepsilon, B_\varepsilon) \Leftrightarrow \exists K \in \mathbb{R}^{m\times n}$ with $(A_\varepsilon + B_\varepsilon K)\mathcal{V} \subseteq \mathcal{V}$

$\mathcal{V} \supseteq E_\varepsilon \Leftrightarrow (v_1, v_2, \cdots, v_k$ a basis of $\mathcal{V})$

$$\exists \psi_1, \psi_2, \cdots, \psi_k \in \mathbb{R} \text{ with } E_\varepsilon = \sum_{\xi=1}^{k} \psi_\xi v_\xi$$

Let now $p(s) = p_0 + p_1 s + \cdots + p_k s^k$ be the annihilating polynomial of $\mathcal{V}$. We must have that

$$p_k(A_\varepsilon + B_\varepsilon K)^k E_\varepsilon + p_{k-1}(A_\varepsilon + B_\varepsilon K)^{k-1} E_\varepsilon + \cdots + p_1(A_\varepsilon + B_\varepsilon K)E_\varepsilon + p_0 E_\varepsilon = 0_{n\times 1} \quad \text{(II.10)}$$

Suppose now that $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the roots of $p(s)$ and $v_1, v_2, \cdots, v_k$ is a particular basis of $\mathcal{V}$, its eigenvectors.

$$(A_\varepsilon + B_\varepsilon K)v_\xi = \lambda_\xi v_\xi \quad \text{(II.11)}$$

Equation (II.10) becomes

$$\sum_{\zeta=0}^{k} p_\zeta (A_\varepsilon + B_\varepsilon K)^\zeta \sum_{\xi=1}^{k} \psi_\xi v_\xi = 0_{n\times 1} \quad \text{(II.12)}$$

Taking into account (II.11)

$$\sum_{\zeta=0}^{k} p_\zeta \sum_{\xi=1}^{k} \psi_\xi \lambda_\xi^\zeta v_\xi = 0_{n\times 1} \quad \text{(II.13)}$$

For the row q of (II.13) one obtains

$$\sum_{\zeta=0}^{k} p_\zeta \sum_{\xi=1}^{k} \psi_\xi \lambda_\xi^{\zeta+q-1} = 0 \Leftrightarrow \sum_{\zeta=0}^{k} p_\zeta e_{\zeta+q} = 0$$

As the above equation holds for every q,

$$\left.\begin{array}{l} \mathcal{V} \in \Im(A_\varepsilon, B_\varepsilon) \\ \mathcal{V} \supseteq E_\varepsilon \\ \dim(\mathcal{V}) = k \end{array}\right\} \Leftrightarrow p(s) \in \mathcal{P}(\mathcal{H}_k(E_\varepsilon)) \quad \textit{e.g.}$$

Examples

$a)\ E_\varepsilon^t = [1 \ \ -2 \ \ 3 \ \ -4 \ \ 5 \ \ -6]$

$b)\ E_\varepsilon^t = [1 \ \ 2 \ \ -3 \ \ 4 \ \ 5 \ \ -6]$

Example a.

$$\mathcal{H}_0 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \\ -6 \end{bmatrix}, \quad \mathcal{H}_1 = \begin{bmatrix} 1 & -2 \\ -2 & 3 \\ 3 & -4 \\ -4 & 5 \\ 5 & -6 \end{bmatrix}, \quad \mathcal{H}_2 = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 3 & -4 & 5 \\ -4 & 5 & -6 \end{bmatrix}$$

$$\mathcal{H}_3 = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -2 & 3 & -4 & 5 \\ 3 & -4 & 5 & -6 \end{bmatrix}, \quad \mathcal{H}_4 = \begin{bmatrix} 1 & -2 & 3 & -4 & 5 \\ -2 & 3 & -4 & 5 & -6 \end{bmatrix}$$

$$\ker(\mathcal{H}_1) = \varnothing, \ \ker(\mathcal{H}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \ker(\mathcal{H}_3) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\ker(\mathcal{H}_4) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right), \mathcal{H}_5 = \mathcal{H}_0^t$$

$$\ker(\mathcal{H}_5) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

There is only one $\mathcal{V} \in \Im(A_\varepsilon, B_\varepsilon)$ of dimension two covering E_ε. Its annihilating polynomial is $\mathcal{P}(\mathcal{H}_2) = p(s) = s^2 + 2s + 1$. The roots of this polynomial are:

$$\rho_1 = \rho_2 = -1. \qquad \text{(II.14)}$$

The annihilating polynomials of the $A \bmod B$ invariant subspaces of dimension 3 covering E_ε are given by the relation:

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_3) = p_0(s) + \mu_1 p_1(s) \\ &p_0(s) = s^3 - 3s - 2,\ p_1(s) = s^2 + 2s + 1 \end{aligned} \qquad \text{(II.15)}$$

The annihilating polynomials of the $A \bmod B$ invariant subspaces of dimension 4 covering E_ε are given by the relation:

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_4) = q_0(s) + \mu_1 q_1(s) + \mu_2 q_2(s) \\ &q_0(s) = s^4 + 4s + 3,\ q_1(s) = s^3 - 3s - 2, \\ &q_2(s) = s^2 + 2s + 1 \end{aligned} \qquad \text{(II.16)}$$

The annihilating polynomials of the $A \bmod B$ invariant subspaces of dimension 5 covering E_ε are given by the relation:

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_5) = \\ &q_0(s) + \mu_1 q_1(s) + \mu_2 q_2(s) + \mu_3 q_3(s) + \mu_4 q_4(s) \\ &q_0(s) = s^5 + 6,\ q_1(s) = s^4 - 5, \\ &q_2(s) = s^3 + 4, q_3(s) = s^2 - 3, q_4(s) = s + 2 \end{aligned} \qquad \text{(II.17)}$$

Example b

$$\mathcal{H}_0 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \\ 5 \\ -6 \end{bmatrix},\quad \mathcal{H}_1 = \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ -3 & 4 \\ 4 & 5 \\ 5 & -6 \end{bmatrix},\ \mathcal{H}_2 = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 4 \\ -3 & 4 & 5 \\ 4 & 5 & -6 \end{bmatrix}$$

$$\ker(\mathcal{H}_1) = \varnothing, \ker(\mathcal{H}_2) = \varnothing$$

$$\mathcal{H}_3 = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & -3 & 4 & 5 \\ -3 & 4 & 5 & -6 \end{bmatrix}, \ker(\mathcal{H}_3) = \begin{bmatrix} -19/6 \\ -2/3 \\ -1/6 \\ 1 \end{bmatrix}$$

$$\mathcal{H}_4 = \begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 2 & -3 & 4 & 5 & -6 \end{bmatrix},\ \mathcal{H}_5 = \begin{bmatrix} 1 & 2 & -3 & 4 & 5 & -6 \end{bmatrix}$$

$$\ker(\mathcal{H}_4) = \text{Span}\left(\begin{bmatrix} 1/7 \\ 10/7 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -22/7 \\ -3/7 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/7 \\ -16/7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\ker(\mathcal{H}_5) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

There is only one $\mathcal{V} \in \Im(A_\varepsilon, B_\varepsilon)$ of dimension three covering E_ε. Its annihilating polynomial is $\mathcal{P}(\mathcal{H}_3) = p(s) = s^3 - \frac{1}{6}s^2 - \frac{2}{3}s - \frac{19}{6}$. The roots of this polynomial are:

$$\rho_1 = 1.6821, \rho_{2,3} = -0.7577 \pm 1.1438i. \qquad \text{(II.18)}$$

The annihilating polynomials of the $A \bmod B$ invariant subspaces of dimension 4 covering E_ε are given by the relation:

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_4) = p_0(s) + \mu_1 p_1(s) + \mu_2 p_2(s) \\ &p_0(s) = s^4 - \frac{16}{7}s - \frac{3}{7},\ p_1(s) = s^3 - \frac{3}{7}s - \frac{22}{7} \\ &p_2(s) = s^2 + \frac{10}{7}s + \frac{1}{7} \end{aligned} \qquad \text{(II.19)}$$

The annihilating polynomials of the $A \bmod B$ invariant subspaces of dimension 5 covering E_ε are given by the relation:

$$\begin{aligned} &\mathcal{P}(\mathcal{H}_5) = \\ &q_0(s) + \mu_1 q_1(s) + \mu_2 q_2(s) + \mu_3 q_3(s) + \mu_4 q_4(s) \\ &q_0(s) = s^5 + 6,\ q_1(s) = s^4 - 5, \\ &q_2(s) = s^3 - 4, q_3(s) = s^2 + 3, q_4(s) = s - 2 \end{aligned} \qquad \text{(II.20)}$$

Corollary I.1

The minimal dimension k, of an $A_\varepsilon \bmod B_\varepsilon$ invariant subspace covering the vector υ, is given by the relation:

$$\ker(\mathcal{H}_{k-1}(\upsilon)) = \varnothing, \ker(\mathcal{H}_k(\upsilon)) \neq \varnothing$$

Corollary I.2

The minimal dimension k, of an $A_\varepsilon \bmod B_\varepsilon$ invariant subspace covering the vectors $\upsilon_1, \upsilon_2, \ldots, \upsilon_r$, is given by the relation:

$$\ker\left(\begin{bmatrix} \mathcal{H}_{k-1}(\upsilon_1) \\ \vdots \\ \mathcal{H}_{k-1}(\upsilon_r) \end{bmatrix} \right) = \varnothing, \ker\left(\begin{bmatrix} \mathcal{H}_k(\upsilon_1) \\ \vdots \\ \mathcal{H}_k(\upsilon_r) \end{bmatrix} \right) \neq \varnothing$$

example c

$$e1=\begin{bmatrix}5\\-8\\13\\-26\end{bmatrix}, e2=\begin{bmatrix}-1\\5\\-19\\65\end{bmatrix}$$

The annihilating polynomial of the minimal A mod B invariant subspace covering $e1$ is $p(s)=s^2+3s+2$

The annihilating polynomial of the minimal A mod B invariant subspace covering $e2$ is $q(s)=s^2+5s+6$

The smallest common multiple $r(s)$ of $p(s)$, $q(s)$ is $r(s)=s^3+6s^2+11s+6$. This polynomial is the annihilating polynomial of an A mod B invariant subspace of dimention 3 covering span($e1$,$e2$) but

$$z(s)=s^3-\frac{299}{17}s-\frac{390}{17}+\mu\left(s^2+\frac{81}{17}s+\frac{82}{17}\right)$$

is the familly of the annihilating polynomials of the A mod B invariant subspaces of dimention 3 covering span($e1$,$e2$) calculated using corollary I.2

Remark that for $\mu=6, z(s)=r(s)$

III. STABILITY

Find the stable polynomials of a family of linear combinations of polynomials is a problem equivalent to the stabilization by output feedback of systems with one input (or one output). In [1] is given an invariant solution, that is not in terms of the parameters $\mu_1,\mu_2,\cdots,\mu_k$ but in terms of the roots of the polynomial $P(\mathcal{H}_k)$. We recall this solution.

Given a family of polynomials:

$$q(s)=q_0(s)+\sum_{r=1}^{m}\mu_r q_r(s),\quad \mu_r\in\mathbb{R} \tag{III.1}$$

The generalized Bezoutian of the polynomials $q_0(s),q_1(s),\cdots,q_m(s)$ is the function:

$$B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})=\frac{\begin{vmatrix}q_0(\lambda_1) & q_1(\lambda_1) & \cdots & q_m(\lambda_1)\\ q_0(\lambda_2) & q_1(\lambda_2) & \cdots & q_m(\lambda_2)\\ \vdots & \vdots & & \vdots\\ q_0(\lambda_{m+1}) & q_1(\lambda_{m+1}) & \cdots & q_m(\lambda_{m+1})\end{vmatrix}}{\prod_{0<r<t<m+2}(\lambda_r-\lambda_t)} \tag{III.2}$$

It can be interpreted as follows: If $\lambda_1,\lambda_2,\cdots,\lambda_m$ are roots of $q(s)$,then the roots in λ_{m+1} of $B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})$ are the other roots of $q(s)$. The function $B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})$ is the multivariate annihilating polynomial of the family of polynomials. So instead of studying the stability of $q(s)$ one can study the stability of $B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})$, considered as polynomial in the variable λ_{m+1}, for stable $\lambda_1,\lambda_2,\cdots,\lambda_m$.

A third solution of the problem could be given in terms of the coefficients of a stable polynomial.

Consider the polynomial having roots the $\lambda_1,\cdots,\lambda_m$:

$$P(s)=s^m+X_{m-1}s^{m-1}+X_{m-2}s^{m-2}+\cdots+X_1s+X_0 \tag{III.3}$$

As $B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})$ is symmetric it can be written as polynomial in the variable λ_{m+1} with coefficients polynomials of $X_{m-1},X_{m-2},\cdots,X_1,X_0$

$$\begin{aligned}&B(\lambda_{m+1})=Y_{m+1}\lambda_{m+1}^{m+1}+Y_m\lambda_{m+1}^{m}+\cdots+Y_1\lambda_{m+1}+Y_0\\&Y_k=f_k(X_{m-1},X_{m-2},\cdots,X_1,X_0)\end{aligned} \tag{III.4}$$

Finally the polynomials $P(s)$ in (II.3) and $B(\lambda_{m+1})$ in (II.4) must be stable.

Let us now calculate the functions $B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})$ for the previous examples.

Example a:

$$B_3(\lambda_1,\lambda_2)=(\lambda_1+1)^2(\lambda_2+1)^2$$

$$B_4(\lambda_1,\lambda_2,\lambda_3)=(\lambda_1+1)^2(\lambda_2+1)^2(\lambda_3+1)^2$$

$$\begin{aligned}&B_5(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)=-\big(5+4\left(\lambda_5+\lambda_2+\lambda_3+\lambda_4\right)+\\&3\left(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5\right)+\\&2\left(\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_4\lambda_5+\lambda_3\lambda_4\lambda_5\right)+\lambda_2\lambda_3\lambda_4\lambda_5\big)\lambda_1-\\&\big(6+5\left(\lambda_5+\lambda_2+\lambda_3+\lambda_4\right)+\\&4\left(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5\right)+\\&3\left(\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_4\lambda_5+\lambda_3\lambda_4\lambda_5\right)+2\lambda_2\lambda_3\lambda_4\lambda_5\big)\end{aligned}$$

It is obvious that for $\lambda_1\in\mathbb{R}^-, B_3$ is always stable, for $\lambda_1,\lambda_2\in\mathbb{R}^-, B_4$ is always stable. For $B_5(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ the problem is analyzed to the following one. With:

$$\left(\lambda_5+\lambda_2+\lambda_3+\lambda_4\right)=-X_3$$

$$\left(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5\right)=X_2$$

$$\left(\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_4\lambda_5+\lambda_3\lambda_4\lambda_5\right)=-X_1$$

$$\lambda_2\lambda_3\lambda_4\lambda_5=X_0$$

We must have the stability of the polynomials

$$\left(5+4X_3+3X_2+2X_1+X_0\right)s+\left(6+5X_3+4X_2+3X_1+2X_0\right)$$

$$s^4-X_3s^3+X_2s-X_1s+X_0$$

Example b:

$$\begin{aligned}&B_3(\lambda_1,\lambda_2,\lambda_3)=7-\frac{2}{7}\left(\lambda_1+\lambda_2+\lambda_3\right)+\\&4\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right)+\frac{31}{7}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)+\\&\frac{22}{7}\left(\lambda_1^2\lambda_3+\lambda_1^2\lambda_2+\lambda_2^2\lambda_3+\lambda_2^2\lambda_1+\lambda_3^2\lambda_2+\lambda_3^2\lambda_1\right)+4\lambda_1\lambda_2\lambda_3+\end{aligned}$$

$$+\frac{1}{7}\left(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2\right)+\frac{4}{7}\left(\lambda_1^2\lambda_2\lambda_3+\lambda_1\lambda_2^2\lambda_3+\lambda_1\lambda_2\lambda_3^2\right)+$$

$$\frac{10}{7}\left(\lambda_1^2\lambda_2^2\lambda_3+\lambda_1^2\lambda_2\lambda_3^2+\lambda_1\lambda_2^2\lambda_3^2\right)+\lambda_1^2\lambda_2^2\lambda_3^2$$

$$B_3(\lambda_1,\lambda_2,\lambda_3)=b_2\left(\lambda_2,\lambda_3\right)\lambda_1^2+b_1\left(\lambda_2,\lambda_3\right)\lambda_1+b_0\left(\lambda_2,\lambda_3\right)$$

$$b_2\left(\lambda_2,\lambda_3\right)=\frac{31}{7}+\frac{22}{7}\left(\lambda_3+\lambda_2\right)+\frac{1}{7}\left(\lambda_2^2+\lambda_3^2\right)+$$

$$\frac{4}{7}\lambda_2\lambda_3+\frac{10}{7}\left(\lambda_2^2\lambda_3+\lambda_2\lambda_3^2\right)+\lambda_2^2\lambda_3^2$$

$$b_1\left(\lambda_2,\lambda_3\right)=-\frac{2}{7}+4\left(\lambda_2+\lambda_3\right)+\frac{22}{7}\left(\lambda_2^2+\lambda_3^2\right)+4\lambda_2\lambda_3+$$

$$\frac{4}{7}\left(\lambda_2^2\lambda_3+\lambda_2\lambda_3^2\right)+\frac{10}{7}\lambda_2^2\lambda_3^2$$

$$b_0\left(\lambda_2,\lambda_3\right)=7-\frac{2}{7}\left(\lambda_2+\lambda_3\right)+4\lambda_2\lambda_3+\frac{31}{7}\left(\lambda_2^2+\lambda_3^2\right)+$$

$$\frac{22}{7}\left(\lambda_2^2\lambda_3+\lambda_3^2\lambda_2\right)+\frac{1}{7}\left(\lambda_2^2\lambda_3^2\right)$$

The conditions for stability are:

$$b_2\left(\lambda_2,\lambda_3\right)>0, b_1\left(\lambda_2,\lambda_3\right)>0, b_0\left(\lambda_2,\lambda_3\right)>0$$

If now λ_2,λ_3 are the roots of the polynomial s^2-Xs+Y the stability of $B_3(\lambda_1,\lambda_2,\lambda_3)$ is deduced by the set of inequalities

$$31+22X+2Y+X^2+10XY+7Y^2>0$$

$$-2+28X-16Y+22X^2+4XY+10Y^2>0$$

$$28-2X-34Y+31X^2+22XY+Y^2>0$$

$$X<0 \qquad Y>0$$

$$B_5(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)=\left(-5+4\left(\lambda_5+\lambda_2+\lambda_3+\lambda_4\right)+\right.$$

$$3\left(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5\right)+$$

$$\left.2\left(\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_4\lambda_5+\lambda_3\lambda_4\lambda_5\right)-\lambda_2\lambda_3\lambda_4\lambda_5\right)\lambda_1+$$

$$\left(-6-5\left(\lambda_5+\lambda_2+\lambda_3+\lambda_4\right)+\right.$$

$$4\left(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5\right)+$$

$$\left.3\left(\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_4\lambda_5+\lambda_3\lambda_4\lambda_5\right)+2\lambda_2\lambda_3\lambda_4\lambda_5\right)$$

For stability we must have

$$\left(-5+4X+3Y+2Z-W\right)\left(-6-5X+4Y+3Z+2W\right)>0$$

and the stability of the polynomial

$$s^4-Xs^3+Ys^2-Zs+W$$

IV. Questions Of Computer Implementation

To carry out the calculations of this paper we used Matlab 6.5. For the null spaces of the Hankel matrices we used the procedure of Jordan-Gauss elimination and numeric calculation. For the generalized Bezoutians we used symbolic calculations.

The generalized Bezoutian in (III.2) is a symmetric multivariate polynomial. Its degree in each variable is *N-m* with *N* the maximum degree of the univariate polynomials $q_0(s), q_1(s), \cdots, q_m(s)$. This polynomial is a sum of $\left(N-m+1\right)^m$ monomials but thanks to symmetry it depends upon a very small number of parameters as all the monomials with the same distribution of degrees, like x^2yz, xy^2z, xyz^2 have the same coefficient. However the determinant in the numerator is a sum of a big number of monomials between $(m+1)!$ and $(m+1)!(N+1)^{m+1}$. This object is "too large" for Matlab even for small values of *m*, *N*. In this paper we did n' t arrive to exploit the symmetry of the Bezoutian. The calculations are carried out using the Laplace development of the Bezoutian. Remark that we can separate the indices $1,2,\cdots,m,m+1$ in two groups G_1, G_2 with:

$G_1=\left\{\zeta_1,\zeta_2,\cdots,\zeta_l\right\}$ and $G_2=\left\{\xi_1,\xi_2,\cdots,\xi_{m-l+1}\right\}$

$G_1\bigcup G_2=\left\{1,2,\cdots,m+1\right\}, G_1\bigcap G_2=\varnothing$

Then the Bezoutian is written as:

$$B(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})=\frac{\sum sign(w)B_w(\lambda_{\zeta_1}\cdots,\lambda_{\zeta_l})B_{\bar{w}}(\lambda_{\xi_1}\cdots,\lambda_{\xi_{m-l+1}})}{\prod(\lambda_\gamma-\lambda_\delta)}$$ with:

w: a combination of l indices among $1,2,\cdots,m+1$.

$\bar{w}$: the other indices among $1,2,\cdots,m+1$ if we take out *w*.

$sign(w)=(-1)^{i+j}$.

$i=\zeta_1+\zeta_2+\cdots+\zeta_l$.

j = the sum of the indices contained in *w*.

$\gamma\in G_1, \delta\in G_2$

The sumation is over all the combinations of indices *w*.

The product is over all the $\gamma\in G_1$ and $\delta\in G_2$

The functions $B_w(\lambda_{\zeta_1}\cdots,\lambda_{\zeta_l})$ and $B_{\bar{w}}(\lambda_{\xi_1}\cdots,\lambda_{\xi_{m-l+1}})$ are Bezoutians, but of a smaller size. We keep dividing this way up to arriving to implementable objects.

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