On the cancelation of decoupling zeros at $s=\infty$

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***Abstract*— We examine the mechanism of cancelations of decoupling zeros at infinity of a left polyniomial matrix description (left PMD) of a linear multivariale system. It is shown that a theory completly analogus to the classical Rosenbrock-Wolovich theory regarding the cancelation of finite decoupling zeros can be formulated. The whole theory is illustrated by a simple example.**

Index Terms - Linear Multivariable Systems, Polynomial Matrix Models, Decoupling Zeros, Structure at Infinity.

I. Cancelation of decoupling zeros at $s=\infty$ for PMDs

Consider the left PMD $\left[A(s)\in\mathbb{R}[s]^{p\times p}, B(s)\in\mathbb{R}[s]^{p\times m}, I_p, 0_{p,m}\right]$ of a linear multivariable system Σ :

$$A(\rho)\beta(t)=B(\rho)u(t) \tag{1}$$

$$y(t)=\beta(t), \qquad (r=p) \tag{2}$$

which after Laplace transformation with zero initial conditions can be written in Rosenbrock system matrix form

$$\begin{bmatrix} A(s) & B(s) \\ -I_p & 0_{pm}\end{bmatrix}\begin{bmatrix}\beta(s)\\ -u(s)\end{bmatrix}=\begin{bmatrix}0_{r1}\\ -y(s)\end{bmatrix}$$

Σ can also be written in *normalized form* as

$$\begin{bmatrix} A(s) & B(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp}\end{bmatrix}\begin{bmatrix}\beta(s)\\ -u(s)\\ y(s)\end{bmatrix}=\begin{bmatrix}0_{pm}\\ 0_{pm}\\ I_m\end{bmatrix}u(s)$$

$$y(s)=\begin{bmatrix}0_{pp} & 0_{pm} & I_p\end{bmatrix}\begin{bmatrix}\beta(s)\\ -u(s)\\ y(s)\end{bmatrix}$$

Definition 1: The *system poles* in $\mathbb{C}\cup\{\infty\}$ of $\Sigma=[A(s), B(s), I_p, 0_{p,m}]$ are the *zeros* of the *normalized denominator*

$$\mathcal{T}(s)=\begin{bmatrix} A(s) & B(s) & 0_{p,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp}\end{bmatrix}\in\mathbf{R}[s]^{(2p+m)\times(2p+m)}$$

The *generalized order* f of Σ is the total number of the *system poles* of Σ in $\mathbb{C}\cup\{\infty\}$ or equivalently the total number of *zeros* of $\mathcal{T}(s)$ in $\mathbb{C}\cup\{\infty\}$ (where multiplicities and orders of the zeros in $\mathbb{C}\cup\{\infty\}$ of $\mathcal{T}(s)$ are accounted for). ■

Since the following row and column operations can be accomplished by pre or post multiplying $\mathcal{T}(s)$ by constant non-singular matrices involving only the identity matrix, and so both $\mathbf{R}[s]$ and $\mathbf{R}_{pr}(s)$ - unimodular

$$\mathcal{T}(s)=\begin{bmatrix} A(s) & B(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp}\end{bmatrix}$$

$$\sim\begin{bmatrix} A(s) & B(s) & 0_{p,m} \\ 0_{pp} & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp}\end{bmatrix}\sim\begin{bmatrix} A(s) & B(s) & 0_{p,m} \\ 0_{mp} & -I_m & 0_{mp} \\ 0_{pp} & 0_{pm} & I_p\end{bmatrix} \tag{3}$$

the *system poles* in $\mathbb{C}\cup\{\infty\}$ of Σ are the zeros in $\mathbb{C}\cup\{\infty\}$ of the matrix

$$F(s):=\begin{bmatrix} A(s) & B(s) \\ 0_{mp} & I_m\end{bmatrix} \tag{4}$$

Notice that the finite poles of Σ are still the finite zeros of $F(s)$ which coincide with the finite zeros of $A(s)$. *The strange thing is that unlike pure generalized state space systems (in which case* $r=n$ and $A(s)=sE-A, E\in\mathbb{R}^{n\times n}, A\in\mathbb{R}^{n\times n}$, $B(s)=B\in\mathbb{R}^{n\times m}$ is a constant matrix and the *the system poles of* Σ at infinity depend only on the matrix pencil $sE-A$) *in this case the system poles of* Σ *at* $s=\infty$ *depend on both the matrices* $A(s)$ **and** $B(s)$.

If $\begin{bmatrix} A(s) & B(s) \\ 0_{mp} & I_m\end{bmatrix}$ has zeros at $s=\infty$ then Σ has *system poles* at $s=\infty$ and it is not *internally proper*. Now some of these *system poles* of Σ at $s=\infty$ may be input (output) (or input-output) decoupling zeros at $s=\infty$ of Σ and may be canceled out when the transfer function matrix $G(s)=A(s)^{-1}B(s)\in\mathbb{R}(s)^{p\times m}$ of Σ is formed.

Definition 2: The *input decoupling zeros (i.d.z.) (output decoupling zeros (o.d.z))* in $\mathbf{C}\cup\{\infty\}$ of Σ are the zeros in $\mathbf{C}\cup\{\infty\}$ of $\begin{bmatrix}\mathcal{T}(s) & \mathcal{U}\end{bmatrix}$ $\left(\begin{bmatrix}\mathcal{T}(s)\\ \mathcal{V}\end{bmatrix}\right)$ and the *input-output decoupling zeros* (*i.o.d.z.*) *in* $\mathbf{C}\cup\{\infty\}$of Σ are the *common* zeros *in* $\mathbf{C}\cup\infty$ of $\begin{bmatrix}\mathcal{T}(s) & \mathcal{U}\end{bmatrix}$and $\begin{bmatrix}\mathcal{T}(s)\\ \mathcal{V}\end{bmatrix}$. The *decoupling zeros* (*d.z.*) *in* $\mathbf{C}\cup\{\infty\}$ of Σ are the elements of the set: {i.d.z. at *in* $\mathbf{C}\cup\{\infty\}$ of Σ} + {o.d.z. *in* $\mathbf{C}\cup\{\infty\}$ of Σ} − {i.o.d. zeros *in* $\mathbf{C}\cup\{\infty\}$of Σ}.

Remark 3: Candidates for *(i.d.z.)* and *(o.d.z)* of Σ are the zeros in $\mathbb{C}\cup\{\infty\}$ of $\mathcal{T}(s)$ i.e. the *system poles of* Σ in $\mathbf{C}\cup\{\infty\}$.

Thus the *input decoupling zeros* of Σ *in* $\mathbb{C}\cup\{\infty\}$of Σ are the zeros *in* $\mathbb{C}\cup\{\infty\}$ of

$$\left[\begin{array}{cc} \mathcal{T}(s) & \mathcal{U} \end{array}\right]=\left[\begin{array}{cccc} A(s) & B(s) & 0_{r,m} & 0_{pm} \\ -I_p(s) & 0_{pm} & I_p & 0_{pm} \\ 0_{mp} & -I_m & 0_{mp} & I_m \end{array}\right]$$

or $\mathbb{R}[s]$ and $\mathbb{R}_{pr}(s)$ $-$equivalently of

$$\left[\begin{array}{ccc} A(s) & B(s) & 0_{mp} \\ -I_p & 0_{pm} & I_p \end{array}\right]$$

or equivalently (adding the last column of the above matrix to the first column, which is an operation of equivalence at infinity) of

$$\left[\begin{array}{ccc} A(s) & B(s) & 0_{mp} \\ 0_{pp} & 0_{pm} & I_p \end{array}\right]$$

or equivalently of

$$\left[\begin{array}{cc} A(s) & B(s) \end{array}\right]$$

Remark 4: So the *input decoupling zeros* of Σ *in* $\mathbb{C}\cup\{\infty\}$ are the zeros *in* $\mathbb{C}\cup\{\infty\}$ of $\left[\begin{array}{cc} A(s) & B(s) \end{array}\right]$.

Notice that if $\left[\begin{array}{cc} A(s) & B(s) \end{array}\right]$ is row proper so that it has no zeros at $s=\infty$, i.e. Σ has no i.d. zeros at $s=\infty$, the matrix $\left[\begin{array}{cc} A(s) & B(s) \\ 0_{mr} & -I_m \end{array}\right]$ might not be row proper (for example when $\left[\begin{array}{cc} A(s) & B(s) \end{array}\right]_h^r=\left[\begin{array}{cc} 0_{pp} & I_m \end{array}\right]$) so that it will have zeros at $s=\infty$ and consequently Σ will have *system poles* at $s=\infty$.

The output decoupling zeros in $\mathbf{C}\cup\{\infty\}$of Σ are the zeros in $\mathbf{C}\cup\{\infty\}$ of

$$\left[\begin{array}{cc} A(s) & B(s) \\ -I_p & 0_{pm} \\ 0_{mp} & -I_m \end{array}\right] \tag{5}$$

Let that (5) has zeros at infinity, and let $\left[\begin{array}{cc} U_1(s) & U_2(s) \\ U_3(s) & U_4(s) \end{array}\right]$ unimodular and such that

$$\left[\begin{array}{cc} A(s) & B(s) \\ -I_p & 0_{pm} \\ 0_{mp} & -I_m \end{array}\right]\left[\begin{array}{cc} U_1(s) & U_2(s) \\ U_3(s) & U_4(s) \end{array}\right] = \left[\begin{array}{cc} A_2(s) & B_2(s) \\ -U_1(s) & -U_2(s) \\ -U_3(s) & -U_4(s) \end{array}\right]$$

is column proper, i.e. it has no zeros at infinity, so that

$$\begin{aligned} &\left[\begin{array}{cc} A(s) & B(s) \\ -I_p & 0_{pm} \\ 0_{mp} & -I_m \end{array}\right] \\ =&\left[\begin{array}{cc} A_2(s) & B_2(s) \\ -U_1(s) & -U_2(s) \\ -U_3(s) & -U_4(s) \end{array}\right]\left[\begin{array}{cc} U_1(s) & U_2(s) \\ U_3(s) & U_4(s) \end{array}\right]^{-1} \\ =&\left[\begin{array}{cc} A_2(s) & B_2(s) \\ -U_1(s) & -U_2(s) \\ -U_3(s) & -U_4(s) \end{array}\right]\widehat{U}_R(s) \end{aligned}$$

Definition 5: A system $\widehat{\Sigma}=\left[\widehat{A}(s), \widehat{B}(s), I_p, 0_{p,m}\right]$ which has no input and no output decoupling zeros in $\mathbf{C}\cup\{\infty\}$ is called strongly *irreducible*

Given a system $\Sigma=[A(s), B(s), I_p, 0_{p,m}]$ the next corollary gives a relation between **(i)** the *set of system poles* of Σ in $\mathbf{C}\cup\{\infty\}$ i.e *set of zeros* in $\mathbf{C}\cup\{\infty\}$ of $\mathcal{T}(s)$,(ii) the *set of poles* in $\mathbf{C}\cup\{\infty\}$ of the transfer function $G(s)$ of Σ and (iii) the *set of decoupling zeros* in $\mathbf{C}\cup\{\infty\}$ of Σ.

Corollary 6:

$$\begin{aligned} &\{\text{set of zeros } \textit{in } \mathbf{C}\cup\{\infty\} \text{ of } \mathcal{T}(s)\} \\ \equiv\ &\{\text{set of poles in } \mathbf{C}\cup\{\infty\} \text{ of } G(s)\} \\ &\cup\{\text{set of decoupling zeros in } \mathbf{C}\cup\{\infty\} \text{ of } \Sigma\ \} \end{aligned} \tag{6}$$

The above set relation gives rise to the equation

$$\begin{aligned} f\ :&=\#\text{ of zeros in } \mathbf{C}\cup\{\infty\} \text{ of } \mathcal{T}(s) \\ =&\ \#\text{ of poles in } \mathbf{C}\cup\{\infty\} \text{ of } G(s) \\ &+\#\text{ decoupling zeros in } \mathbf{C}\cup\{\infty\} \text{ of } \Sigma \end{aligned} \tag{7}$$

where the symbol # means the “total number” and where multiplicities and orders of a pole or zero at $s=\infty$ are accounted for.

Remark 7: Using the fact that the total number of zeros in $\mathbf{C}\cup\{\infty\}$ of a *square polynomial* matrix is equal to the total number of its poles in $\mathbf{C}\cup\{\infty\}$ which for a polynomial [13] matrix coinsides with its McMillan degree, equation (7) can be written as

$$\begin{aligned} f=\delta_M[\mathcal{T}(s)]=\delta_M(G(s)) \\ +\#\text{ decoupling zeros in } \mathbf{C}\cup\{\infty\} \text{ of } \Sigma \end{aligned} \tag{8}$$

Eqn. (7) gives rise to the inequality

$$\begin{aligned} f:=\#\text{ of zeros in } \mathbf{C}\cup\{\infty\} \text{ of } [\mathcal{T}(s)] \\ =\delta_M[\mathcal{T}(s)] \\ \geq\ \#\text{ of poles in } \mathbf{C}\cup\{\infty\} \text{ of } G(s) \\ =:\delta_M(G(s)) \end{aligned} \tag{9}$$

If a system $\widehat{\Sigma}=\left[\widehat{A}(s), \widehat{B}(s), I_p, 0_{p,m}\right]$ is strongly irreducible i.e. if $\widehat{\Sigma}$ has no i.d. and no o.d. zeros in $\mathbf{C}\cup\{\infty\}$ then from Definition 2: [# decoupling zeros of $\widehat{\Sigma}$ in $\mathbf{C}\cup\{\infty\}$]= 0 and from (7) the generalized order $\widehat{f}$ of $\widehat{\Sigma}$ takes its least value:

$$\widehat{f}=\delta_M[\mathcal{T}(s)]=\sum_{i=1}^{k} q_i(\mathcal{T}(s))=\delta_M(G(s)) \tag{10}$$

In such a case the strongly irreducible system $\widehat{\Sigma}=\left[\widehat{A}(s), \widehat{B}(s), I_p, 0_{p,m}\right]$ has the *least generalized order* $\widehat{f}$ among the generalized orders of all systems Σ which give rise to $G(s)$. As indicated by (10) the *least* generalized order $\widehat{f}$ of $\widehat{\Sigma}$ can then be determined directly as the McMillan degree of the transfer function matrix $G(s)$.

Definition 5 together with the above discussion give rise to

Theorem 8: A system $\widehat{\Sigma} = \left[\widehat{A}(s), \widehat{B}(s), I_p, 0_{p,m}\right]$ with transfer function matrix $G(s)$ is strongly *irreducible* iff

$$\widehat{f} := \#\text{ of zeros at } s=\infty \text{ of } \mathcal{T}(s) = \delta_M[\mathcal{T}(s)]$$

$$= \sum_{i=1}^{k} q_i(\mathcal{T}(s)) = \delta_M(G(s)) \tag{11}$$

A. Extraction of i.d. zeros at infinity for left PMDs

Let $\begin{bmatrix} A(s) & B(s) \end{bmatrix}$ have zeros at $s=\infty$ and assume that $U_L(s) \in \mathbf{R}[s]^{p\times p}$ is unimodular and such that

$$U_L(s)\begin{bmatrix} A(s) & B(s) \end{bmatrix} =: \begin{bmatrix} \widehat{A}(s) & \widehat{B}(s) \end{bmatrix} \tag{12}$$

is *row proper* so that $\begin{bmatrix} \widehat{A}(s) & \widehat{B}(s) \end{bmatrix}$ has no zeros at $s=\infty$. Let $\widehat{U}_L(s) = U_L(s)^{-1} \in \mathbf{R}[s]^{p\times p}$ and write (12) as

$$\begin{bmatrix} A(s) & B(s) \end{bmatrix} = \widehat{U}_L(s)\begin{bmatrix} \widehat{A}(s) & \widehat{B}(s) \end{bmatrix} \tag{13}$$

The unimodular matrix $\widehat{U}_L(s)$ in (13) can be viewed as a 'greatest common left divisor at $s=\infty$ 'of $A(s)$ and $B(s)$ so that the pair $\widehat{A}(s), \widehat{B}(s)$ are 'left coprime at $s=\infty$ '. Now the transfer function matrix $G(s)$ of Σ is

$$G(s) = \begin{bmatrix} 0_{pp} & 0_{pm} & I_p \end{bmatrix}\begin{bmatrix} A(s) & B(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp} \end{bmatrix}^{-1}\begin{bmatrix} 0_{pm} \\ 0_{pm} \\ I_m \end{bmatrix}$$

Since we can write

$$\begin{bmatrix} A(s) & B(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp} \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} \widehat{U}_L(s) & 0_{pp} & 0_{pm} \\ 0_{pp} & I_p & 0_{pm} \\ 0_{mp} & 0_{mp} & I_m \end{bmatrix}$$

$$\times \begin{bmatrix} \widehat{A}(s) & \widehat{B}(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp} \end{bmatrix}$$

$$\begin{bmatrix} 0_{pm} \\ 0_{pm} \\ I_m \end{bmatrix} = \begin{bmatrix} \widehat{U}_L(s) & 0_{pp} & 0_{pm} \\ 0_{pp} & I_p & 0_{pm} \\ 0_{mp} & 0_{mp} & I_m \end{bmatrix}\begin{bmatrix} 0_{pm} \\ 0_{pm} \\ I_m \end{bmatrix} \tag{15}$$

we have that

$$G(s)$$

$$= \begin{bmatrix} 0_{pp} & 0_{pm} & I_p \end{bmatrix}\begin{bmatrix} \widehat{A}(s) & \widehat{B}(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \widehat{U}_L(s) & 0_{pp} & 0_{pm} \\ 0_{pp} & I_p & 0_{pm} \\ 0_{mp} & 0_{mp} & I_m \end{bmatrix}^{-1}\begin{bmatrix} \widehat{U}_L(s) & 0_{pp} & 0_{pm} \\ 0_{pp} & I_p & 0_{pm} \\ 0_{mp} & 0_{mp} & I_m \end{bmatrix}$$

$$\begin{bmatrix} 0_{pm} \\ 0_{pm} \\ I_m \end{bmatrix} u(s)$$

$$= \begin{bmatrix} 0_{pp} & 0_{pm} & I_p \end{bmatrix}\begin{bmatrix} \widehat{A}(s) & \widehat{B}(s) & 0_{r,m} \\ -I_p(s) & 0_{pm} & I_p \\ 0_{mp} & -I_m & 0_{mp} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 0_{pm} \\ 0_{pm} \\ I_m \end{bmatrix} u(s)$$

$$= \begin{bmatrix} 0_{pp} & 0_{pm} & I_p \end{bmatrix}\begin{bmatrix} \widehat{A}(s)^{-1} & 0_{pp} & \widehat{A}(s)^{-1}\widehat{B}(s) \\ 0_{mp} & 0_{mp} & -I_m \\ \widehat{A}(s)^{-1} & I_p & \widehat{A}(s)^{-1}\widehat{B}(s) \end{bmatrix}$$

$$\begin{bmatrix} 0_{pm} \\ 0_{pm} \\ I_m \end{bmatrix} u(s)$$

$$= \begin{bmatrix} 0_{pp} & 0_{pm} & I_p \end{bmatrix}\begin{bmatrix} \widehat{A}(s)^{-1}\widehat{B}(s) \\ -I_m \\ \widehat{A}(s)^{-1}\widehat{B}(s) \end{bmatrix} u(s)$$

$$= \widehat{A}(s)^{-1}\widehat{B}(s)u(s)$$

In general, after such a cancelation of *system poles* of Σ at $s=\infty$ with *i.* or *o. decoupling zeros* at $s=\infty$ of Σ, $G(s)$ need no be proper. Simply the new system $\widehat{\Sigma} = \left[\widehat{A}(s),\ \widehat{B}(s), I_p, 0_{pm}\right]$ will be irreducible at $s=\infty$. If $G(s)$ is non-proper then the poles of $G(s)$ at $s=\infty$ will be zeros at $s=\infty$ of $\mathcal{T}(s)$ which are not *i.d.zeros* or *o.d.zeros* at $s=\infty$ of $\widehat{\Sigma}$ and thus they are not 'canceled out' when $G(s)$ is formed. If $G(s)$ is proper and $\mathcal{T}(s)$ has zeros at $s=\infty$ which are i. or *o. decoupling zeros at* $s=\infty$ of Σ then $\mathcal{T}(s)^{-1}$ will be non-proper and Σ is *externaly proper* but not *internaly proper.* So the candidates for decoupling zeros at infinity are the zeros at infinity of $\mathcal{T}(s)$.

If a system is irreducible at $s=\infty$ then the Laurent expansion of its transfer function matrix $G(s)$ 'starts' from the degree of the pole at infinity of $G(s)$ which has maximum value, i.e. from $q_1(G(s))$ in the Smith-Mcmillan form at $s=\infty : S^{\infty}_{G(s)}$of $G(s)$, this number will be equal to the order of the zero at $s=\infty$ of maximum value of $\mathcal{T}(s)$ i.e. of $\widehat{q}_{2p+m}(\mathcal{T}(s))$ in the Smith-Mcmillan form at $s=\infty : S^{\infty}_{\mathcal{T}(s)}$ of $\mathcal{T}(s)$.

Example 9: Consider a system Σ described by the PMD: $A(s) = \begin{bmatrix} s+1 & s^3 \\ 0 & s+1 \end{bmatrix}$, $B(s) = \begin{bmatrix} s+2 & (s+1)^2 \\ 0 & 1 \end{bmatrix}$, $C(s) = I_2$, $D(s) = 0_{22}$ i.e. with $r=p=m=2$, where $A(s), B(s)$ are left coprime in $\mathbf{C}$, $A(s), B(s)$ are trivially right coprime in $\mathbf{C}$ and $G(s) = A(s)^{-1}B(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{3s^2+3s+1}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \in \mathbf{R}_{pr}(s)^{2\times 2}$ i.e. Σ is *externally proper* but not *internally proper* since the normalized denominator:

$$\mathcal{T}(s) \tag{16}$$

$$= \begin{bmatrix} A(s) & B(s) & 0_{rp} \\ -I_p(s) & 0_{pp} & I_p \\ 0_{mp} & -I_m & 0_{mp} \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc|cc} s+1 & s^3 & s+2 & (s+1)^2 & 0 & 0 \\ 0 & s+1 & 0 & 1 & 0 & 0 \\ - & - & - & - & - & - \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

has *a zero* at $s = \infty$ as it is indicated by

$$S^{\infty}_{\mathcal{T}(s)} = diag\left[s^3, 1, 1, 1, 1, \frac{1}{s}\right]$$

$\left(\tau = r + p + m = 6, \overline{k} = 1, \overline{v} = 5\right), q_1(\mathcal{T}) = 3$, $q_2(\mathcal{T}) = q_3(\mathcal{T}) = q_4(\mathcal{T}) = q_5(\mathcal{T}) = 0, \widehat{q}_6(\mathcal{T}) = 1$, i.e. $\mathcal{T}(s)$ has one *zero* at $s = \infty$ of order $\widehat{q}_6(\mathcal{T}) = 1$ which is a *system pole* of Σ at $s = \infty$. This follows also from the computation of the Smith-McMillan form at $s = \infty$ of

$$F(s) \; : \; = \begin{bmatrix} A(s) & B(s) \\ 0_{mr} & I_m \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & s^3 & s+2 & (s+1)^2 \\ 0 & s+1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since

$$S^{\infty}_{F(s)} = diag\left[s^3, 1, 1, \frac{1}{s}\right]$$

Now since $|\mathcal{T}(s)| = |T(s)| = (s+1)^2$, $n := \deg|T(s)| = 2$ and $\mathcal{T}(s)$ has also one finite zero at $s = -1$ of multiplicity 2, i.e. Σ has 3 *system poles* in $\mathbf{C}\cup\{\infty\}$, two finite ones at $s = -1$ and one at $s = \infty$. Since $S^{\infty}_{[\mathcal{T}(s),\mathcal{U}]} = \left[diag\left[s^3, 1, 1, 1, 1, \frac{1}{s}\right], 0_{62}\right]$, $[\mathcal{T}(s), \mathcal{U}]$ has a *zero* at $s = \infty$ which implies that the *system pole* at $s = \infty$ of Σ is an *input decoupling zero* of Σ at $s = \infty$ which is canceled out when the transfer function matrix $G(s)$ is formed and so it does not appear as a *pole at* $s = \infty$ of $G(s)$ which due to this cancellation is proper. Also since $S^{\infty}_{\begin{bmatrix} \mathcal{T}(s) \\ \mathcal{V} \end{bmatrix}} = \begin{bmatrix} diag\left[s^3, 1, 1, 1, 1, 1\right] \\ 0_{26} \end{bmatrix}$, $\begin{bmatrix} \mathcal{T}(s) \\ \mathcal{V} \end{bmatrix}$ has no zeros at $s = \infty$ and thus Σ has no *output decoupling zeros* at $s = \infty$. The *generalized order* of Σ is $f = n + \sum_{i=\overline{v}+1}^{\tau} \widehat{q}_i(\mathcal{T}) = 2 + 1 = 3 = \delta_M[\mathcal{T}(s)] > \delta_M[G(s)] = n = 2$.

Theorem 10: Consider the system $\sum$ described by the left PMD in 1,2. Let

$$u(t) = v(t) - K\beta(t)$$

where $K \in \mathbf{R}^{m\times p}$ so that the closed loop system is described by

$$[A(\rho) + B(\rho)K]\beta(t) = B(\rho)v(t) \tag{17}$$

Then 17 is *internaly proper* or equivalently

$$F_K(s) := \begin{bmatrix} A(s) + B(s)K & B(s) \\ 0_{mp} & I_m \end{bmatrix}$$

has no zeros at $s = \infty$ iff Σ has no input decoupling zeros at $s = \infty$ or equivalently iff $[A(s), B(s)]$ has no zeros at $s = \infty$.

Proof:

(only if) $\exists K \in \mathbf{R}^{m\times p} : F_K(s)$has no zeros at $s = \infty \Rightarrow [A(s), B(s)]$ has no zeros at $s = \infty$.

We have

$$\begin{bmatrix} A(s) + B(s)K & B(s) \\ 0_{mp} & I_m \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ -K & I_m \end{bmatrix}\begin{bmatrix} I_p & 0 \\ K & I_m \end{bmatrix}$$

therefore $F(s)$ and $F_K(s)$ have the same pole-zero structure at $s = \infty$ and : $F_K(s)$ has no zeros at $s = \infty \Rightarrow F(s)$ has no zeros at $s = \infty$. So as the candidates for decoupling zeros at infinity are the zeros at infinity of $\mathcal{T}(s)$ or equivalently of $F(s)$ and $F(s)$ has no zeros at $s = \infty$ i.e. $\sum$ has no decoupling zeros at infinity or equivalently $[A(s), B(s)]$ has no zeros at $s = \infty$.

(if) $[A(s), B(s)]$ has no zeros at $s = \infty \Rightarrow \exists K \in \mathbf{R}^{m\times p}$: $F_K(s)$ has no zeros at $s = \infty$.

For this proof we will need the following

Lemma 11: There exists $K \in \mathbf{R}^{m\times p}$ such that

$$R(s) := \begin{bmatrix} A(s) & B(s) \\ -K & I_m \end{bmatrix}$$

has no zeros at $s = \infty$ iff $[A(s), B(s)]$ has no zeros at $s = \infty$.

$$[A(s), B(s)] = D(s)^{-1}[N_A(s), N_B(s)]$$

$$R(s) = \begin{bmatrix} D(s) & 0 \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} N_A(s) & N_B(s) \\ -K & I \end{bmatrix}$$

$[A(s), B(s)]$ has no zeros at $s = \infty \Rightarrow rank[N_A(\infty), N_B(\infty)] = p \Rightarrow$there exists $K \in \mathbf{R}^{m\times p}$:

$$rank\begin{bmatrix} N_A(\infty) & N_B(\infty) \\ -K & I \end{bmatrix} = p + m$$

$R(s)$has no zeros at $s = \infty$.

Proof: $\left(\begin{array}{c} R(s) \text{ has no zeros at } s = \infty \\ \Rightarrow [A(s), B(s)] \text{ has no zeros at } s = \infty \end{array}\right)$.

We now go back to the proof of (if) . Let that $[A(s), B(s)]$ has no zeros at $s = \infty$.

$$[A(s), B(s)]$$

Let $K \in \mathbf{R}^{m \times p}$ such that

$$\begin{bmatrix} A(s) & B(s) \\ -K & I_m \end{bmatrix}$$

has no zeros at $s = \infty$. Then from

$$\begin{bmatrix} A(s) & B(s) \\ -K & I_m \end{bmatrix} \begin{bmatrix} I_p & 0 \\ K & I_m \end{bmatrix}$$
$$= \begin{bmatrix} A(s) + B(s) K & B(s) \\ 0 & I_m \end{bmatrix} = F_K(s)$$

it follows that $\begin{bmatrix} A(s) & B(s) \\ -K & I_m \end{bmatrix}$ and $F_K(s)$ have the same pole-zero structure in $\mathbf{C} \cup \{\infty\}$and since $\begin{bmatrix} A(s) & B(s) \\ -K & I_m \end{bmatrix}$ has no zeros at $s = \infty$ so does $F_K(s)$

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